

COMPUTER AIDED GEOMETRIC DESIGN

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Preface

This semester is the 24th time I have taught a course at Brigham Young University titled, “Computer Aided Geometric Design.” When I first taught such a course in 1983, the field was young enough that no textbook covered everything that I wanted to teach, and so these notes evolved.

The field now has matured to the point that several semesters worth of valuable material could be compiled. These notes, admittedly biased towards my own interests, reflect my personal preferences as to which of that material is most beneficial to students in an introductory course.

I welcome anyone who has an interest in studying this fascinating topic to make free use of these notes. I invite feedback on typos and on material that could be written more clearly.

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