Differentiation Rules

The Definition of the Derivative

Recall equivalent definitions

1

The derivative of a function f at a number x_0, $f'(x_0)$, is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided that the limit exists and is finite.

2

The derivative of a function f at a number x_0, is a number a, if there is a function $\varepsilon(x - x_0)$ with the properties

$\lim_{x \to x_0} \varepsilon(x - x_0) = 0$ and $f(x) - f(x_0) = a(x - x_0) + (x - x_0)\varepsilon(x - x_0)$.

Basic Differentiation Rules

1 $D(x) = 1$

The derivative of the function f(x)=x is 1.

2 $D(f + g) = D(f) + D(g)$

3 $D(fg) = D(f)g + fD(g)$

The Product Rule

4 $D(f(g(x))) = D(f)(g(x))D(g)(x)$

$$\Leftrightarrow \frac{d\,f(g(x))}{dx} = f'(g(x))g'(x)$$

The Chain Rule

These are the basic differentiation rules which imply all other differentiation rules for rational algebraic expressions.

Derived Differentiation Rules

5 $$D\left(\frac{f}{g}\right)=\frac{gD(f)-fD(g)}{g^2}$$

The Quotient Rule. Follows from the Product Rule.

6 $$D\left(f^{-1}\right)=\frac{1}{D(f)}$$

Inverse Function Rule. Follows from the Chain Rule.

Special Function Rules

7 $$\frac{d\ln(x)}{dx} = \frac{1}{x}$$

8 $$\frac{dx^r}{dx} = rx^{r-1},\ r \in \mathbb{R}$$

9 $$\frac{d\cos(x)}{dx} = -\sin(x)$$

10 $$\frac{d\tan(x)}{dx} = \frac{1}{\cos^2(x)}$$

11 $$\frac{d\arcsin(x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

12 $$\frac{d\arctan(x)}{dx} = \frac{1}{1+x^2}$$

13 $$\frac{d\,e^x}{dx} = e^x$$

14 $$\frac{da^x}{dx} = a^x\ln(a)$$

15 $$\frac{d\sin(x)}{dx} = \cos(x)$$

16 $$\frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}$$

Differential Calculus

- Notion of a partial derivative
- Suppose that

$$y = f(x_1, x_2)$$

- Two partial derivatives

1) The partial derivative of y with respect to x_1 = $\dfrac{\partial y}{\partial x_1}$

 (where x_2 is treated as a constant)

2) The partial derivative of y with respect to x_2 = $\dfrac{\partial y}{\partial x_2}$

 (where x_2 is treated as a constant)

Curve sketching and analysis

$y = f(x)$ must be continuous at each:

- *critical point*: $\frac{dy}{dx} = 0$ or <u>undefined</u>. And don't forget <u>endpoints</u>
- *local minimum*: $\frac{dy}{dx}$ goes (–,0,+) or (–,und,+) or $\frac{d^2y}{dx^2} > 0$
- *local maximum*: $\frac{dy}{dx}$ goes (+,0,–) or (+,und,–) or $\frac{d^2y}{dx^2} < 0$
- *point of inflection*: concavity changes

$\frac{d^2y}{dx^2}$ goes from (+,0,–), (–,0,+),
(+,und,–), or (–,und,+)

II. Properties of Linear Functions

A linear function can be written as

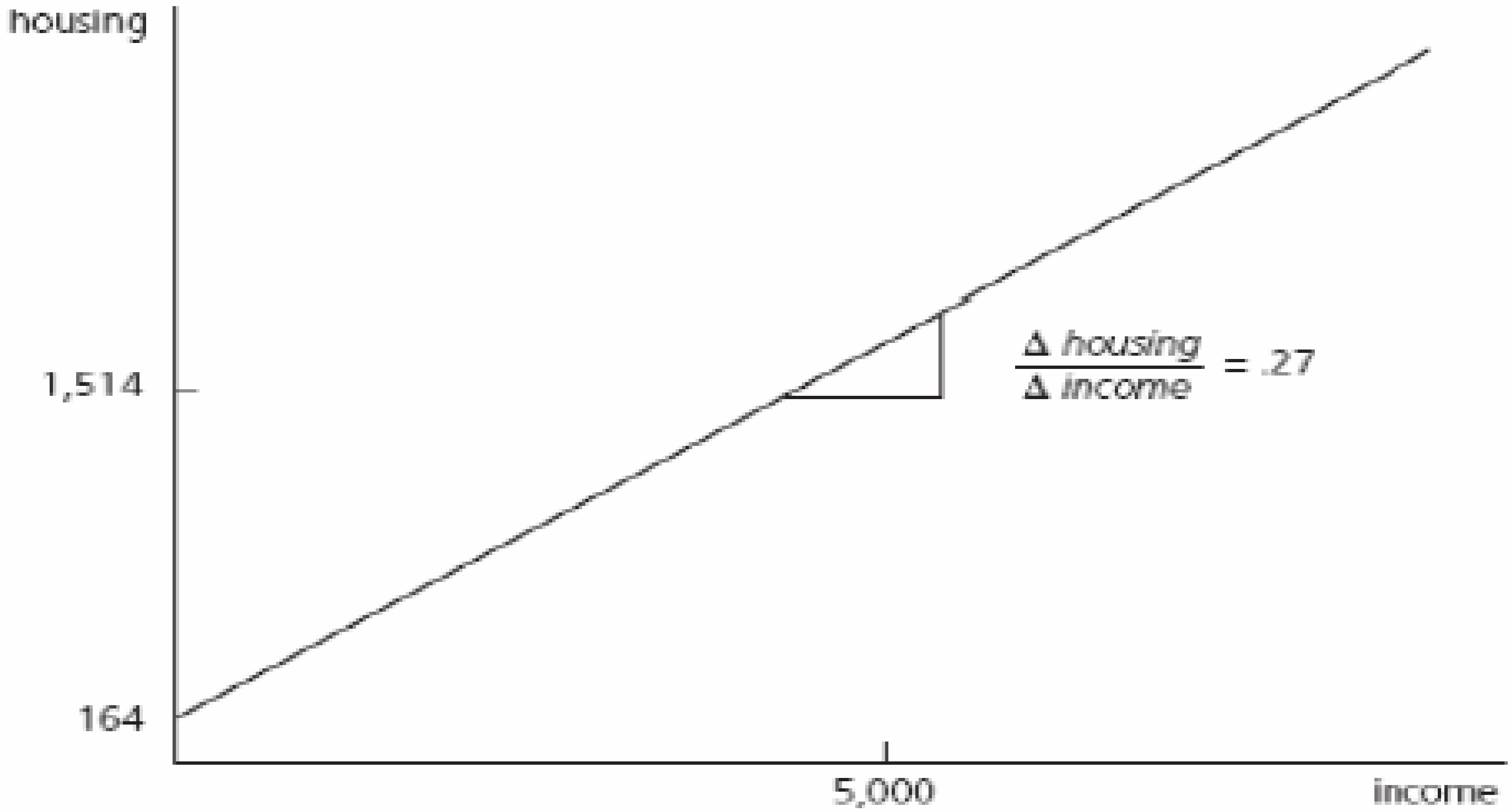
$$y = \beta_0 + \beta_1 x$$

- y and x are variables;
- β_0 and β_1 are parameters;
  - β_0 is called the intercept;
  - β_1 is called the slope.

- We say that y is a linear function of x.

FIGURE A.1

Graph of *housing* = 164 + .27 *income*.

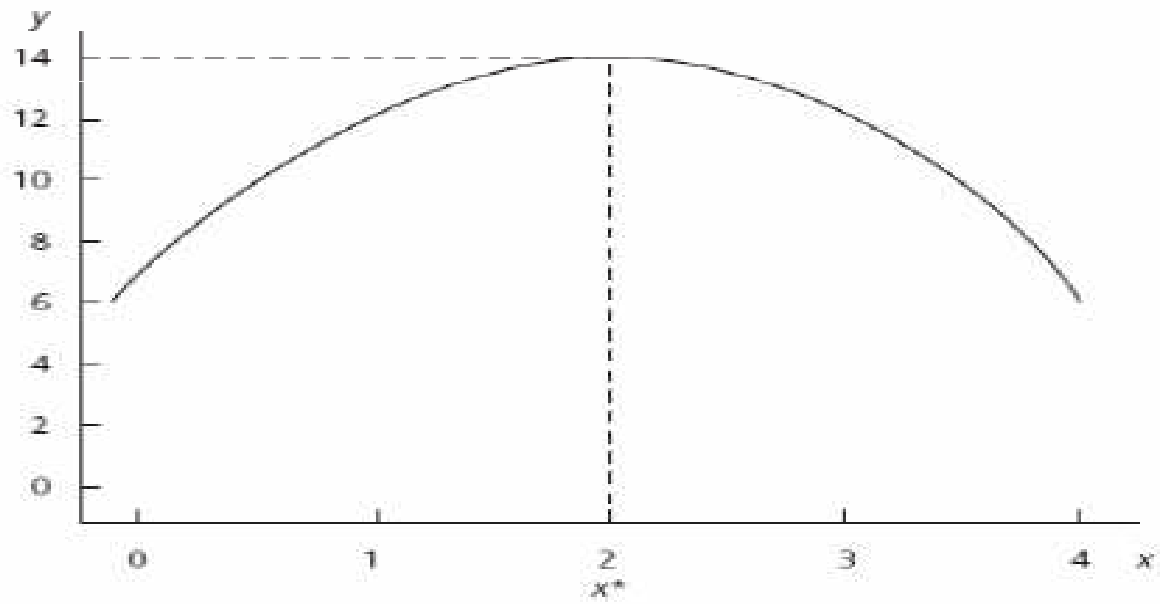


Quadratic functions

$$y = \beta_0 + \beta_1 x + \beta_2 x^2$$

FIGURE A.3

Graph of $y = 6 + 8x - 2x^2$.



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F1

Natural Logarithm

- Properties

1) The log function is defined only for positive values of x (x>0).

   See graph of a log function

   for $0<x<1$ $\log(x) < 0$

   for $x=1$ $\log(x) = 0$

   for $x>1$ $\log(x) > 0$

2) When y= log(x), the effect of x on y never becomes negative.

$$y = \log(x)$$

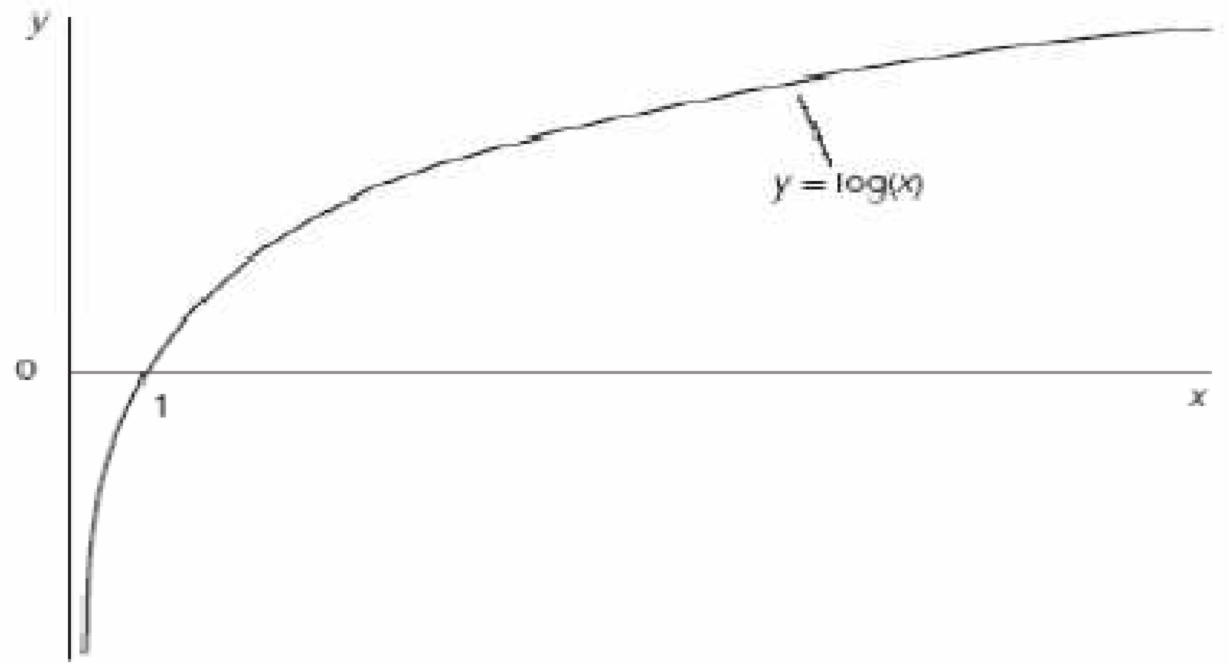
$$\frac{\Delta y}{\Delta x} = \frac{1}{x}$$

- The relationship between x and y displays diminishing returns.
- The slope of the function gets closer and closer to zero as x gets large

Graph: log function

FIGURE A.4

Graph of $y = \log(x)$.



- Some useful algebraic facts:
  - $\log(x_1 \cdot x_2) = \log(x_1) + \log(x_2)$
  - $\log(x_1/x_2) = \log(x_1) - \log(x_2)$
  - $\log(x^c) = c\log(x)$ for any constant c.
  - $\log(1+x) \approx 0$ for $x \approx 0$

Exponential Function

- We write the exponential function as

$$y = \exp(x)$$

  - Other notation can be written as

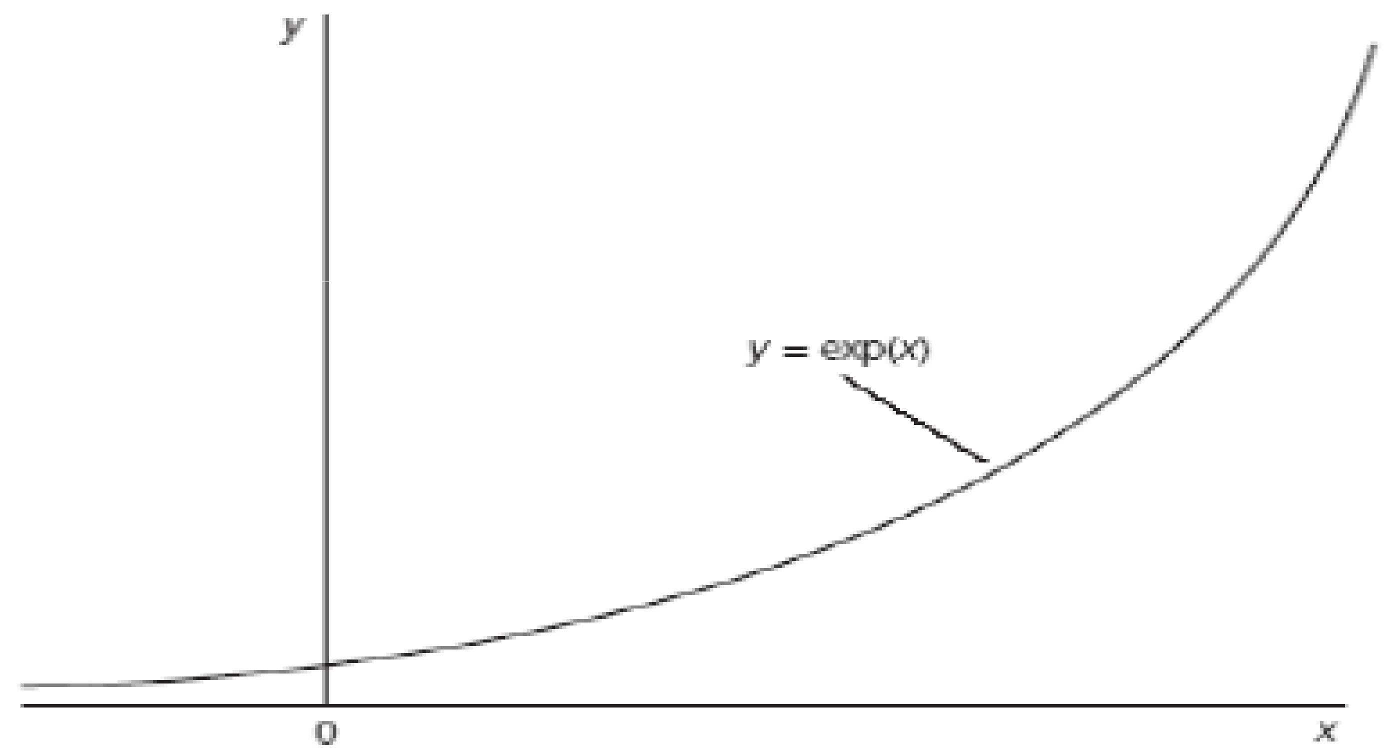
$$y = e^x$$

- Facts
  - exp(0) =1;
  - exp(1) = 2.7183
  - exp(x) is defined for any value of x.
  - exp(x) is always greater than 0

Graph: Exponential Function

FIGURE A.5

Graph of $y = \exp(x)$.



Taylor Series

Brook Taylor was an accomplished musician and painter. He did research in a variety of areas, but is most famous for his development of ideas regarding infinite series.

Brook Taylor
1685 - 1731

Maclaurin Series:

(generated by f at $x = 0$)

$$P(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

If we want to center the series (and it's graph) at some point other than zero, we get the Taylor Series:

Taylor Series:

(generated by f at $x = a$)

$$P(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$\rightarrow$

example: $y = \cos x$

$$f(x) = \cos x \qquad f(0) = 1$$

$$f'(x) = -\sin x \qquad f'(0) = 0$$

$$f''(x) = -\cos x \qquad f''(0) = -1$$

$$f'''(x) = \sin x \qquad f'''(0) = 0$$

$$f^{(4)}(x) = \cos x \qquad f^{(4)}(0) = 1$$

$$P(x) = 1 + 0x - \frac{1x^2}{2!} + \frac{0x^3}{3!} + \frac{1x^4}{4!} + \frac{0x^5}{5!} - \frac{1x^6}{6!} + \cdots$$

$$P(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} \cdots$$

$\rightarrow$

example: $y = \cos(2x)$

Rather than start from scratch, we can use the function that we already know:

$$P(x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \frac{(2x)^8}{8!} - \frac{(2x)^{10}}{10!} \quad \cdots$$

$\rightarrow$

example: $y = \cos(x)$ at $x = \dfrac{\pi}{2}$

$$f(x) = \cos x \quad f\left(\frac{\pi}{2}\right) = 0$$

$$f'(x) = -\sin x \quad f'\left(\frac{\pi}{2}\right) = -1$$

$$f''(x) = -\cos x \quad f''\left(\frac{\pi}{2}\right) = 0$$

$$f'''(x) = \sin x \quad f'''\left(\frac{\pi}{2}\right) = 1$$

$$f^{(4)}(x) = \cos x \quad f^{(4)}\left(\frac{\pi}{2}\right) = 0$$

$$P(x) = 0 - 1\left(x - \frac{\pi}{2}\right) + \frac{0}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{3!}\left(x - \frac{\pi}{2}\right)^3 + \cdots$$

$$P(x) = -\left(x - \frac{\pi}{2}\right) + \frac{\left(x - \frac{\pi}{2}\right)^3}{3!} - \frac{\left(x - \frac{\pi}{2}\right)^5}{5!} + \cdots$$

→

Example:

To get the *cos*(x) for small x:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

If x=0.5

cos(0.5) =1-0.125+0.0026041-0.0000127+ …

=0.877582

From the supporting theory, for this series, the error is no greater than the first omitted term.

$$\therefore \quad \frac{x^8}{8!} \quad for \quad x = 0.5 \quad = 0.0000001$$

$\sin(x)$

$$P(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$f^{(n)}(x)$	$f^{(n)}(0)$
$\sin(x)$	0
$\cos(x)$	1
$-\sin(x)$	0
$-\cos(x)$	-1
$\sin(x)$	0

$$\sin(x) = 0 + 1x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots$$

Both sides are odd functions.

Sin (0) = 0 for both sides.

→

$$\frac{1}{1+x^2}$$

If we start with this function: $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \cdots$

and substitute x^2 for x , we get: $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 + \cdots$

This is a geometric series with $a = 1$ and $r = -x^2$.

If we integrate both sides:

$$\int \frac{1}{1+x^2}\,dx = \int 1 - x^2 + x^4 - x^6 + x^8 + \cdots\,dx$$

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

This looks the same as the series for sin (x), but without the factorials.

→

$\ln(1+x)$

$$P(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$f^{(n)}(x)$	$f^{(n)}(0)$
$\ln(1+x)$	0
$(1+x)^{-1}$	1
$-(1+x)^{-2}$	-1
$2(1+x)^{-3}$	2
$-6(1-x)^{-4}$	$-6=-3!$

$$\ln(1+x) = 0 + 1x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-3!}{4!}x^4 + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

→

e^x

$$P(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$f^{(n)}(x)$	$f^{(n)}(0)$
e^x	1
e^x	1
e^x	1
e^x	1
e^x	1

$$e^x = 1 + 1x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

→

Taylor Series

If the function f is "smooth" at $x = a$, then it can be approximated by the n^{th} degree polynomial

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Maclaurin Series

A Taylor Series about $x = 0$ is called Maclaurin.

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

Taylor's Theorem: Error Analysis for Series

Taylor series are used to estimate the value of functions (at least theoretically - now days we can usually use the calculator or computer to calculate directly.)

An estimate is only useful if we have an idea of how accurate the estimate is.

When we use part of a Taylor series to estimate the value of a function, the end of the series that we do not use is called the remainder. If we know the size of the remainder, then we know how close our estimate is.

$\rightarrow$

For a geometric series, this is easy:

ex. 2: Use $1+x^2+x^4+x^6$ to approximate $\dfrac{1}{1-x^2}$ over $(-1,1)$.

Since the truncated part of the series is: $x^8+x^{10}+x^{12}+\cdots$,

the truncation error is $\left|x^8+x^{10}+x^{12}+\cdots\right|$, which is $\dfrac{x^8}{1-x^2}$.

When you "truncate" a number, you drop off the end.

Of course this is also trivial, because we have a formula that allows us to calculate the sum of a geometric series directly.

→

Taylor's Theorem with Remainder

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

Lagrange Form of the Remainder

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Remainder after partial sum S_n where c is between a and x.

→

Lagrange Form of the Remainder

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Remainder after partial sum S_n where c is between a and x.

This is also called the remainder of order n or the error term.

Note that this looks just like the next term in the series, but "a" has been replaced by the number "c" in $f^{(n+1)}(c)$.

This seems kind of vague, since we don't know the value of c, but we can sometimes find a maximum value for $f^{(n+1)}(c)$.

→

Lagrange Form of the Remainder

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

If M is the maximum value of $f^{(n+1)}(x)$ on the interval between a and x, then:

$$\left|R_n(x)\right| \le \frac{M}{(n+1)!}\left|x-a\right|^{n+1}$$

We will call this the Remainder Estimation Theorem.

→

ex. 2: Prove that $\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$, which is the Taylor series for $\sin x$, converges for all real x.

Since the maximum value of $\sin x$ or any of it's derivatives is 1, for all real x, $M = 1$.

$$\therefore \left|R_n(x)\right| \le \frac{1}{(n+1)!}\left|x-0\right|^{n+1} = \frac{\left|x\right|^{n+1}}{(n+1)!}$$

Remainder Estimation Theorem

$$\left|R_n(x)\right| \le \frac{M}{(n+1)!}\left|x-a\right|^{n+1}$$

$$\lim_{n\to\infty}\frac{\left|x\right|^{n+1}}{(n+1)!} = 0$$

so the series converges.

→

ex. 5: Find the Lagrange Error Bound when $x-\frac{x^2}{2}$ is used to approximate $\ln(1+x)$ and $|x| \le 0.1$.

$$f(x)=\ln(1+x)$$

$$f(x)=0+x-\frac{x^2}{2}+R_2(x)$$

$$f'(x)=(1+x)^{-1}$$

On the interval $[-.1,.1]$, $\frac{2}{(1+x)^3}$ decreases, so its maximum value occurs at the left end-point.

$$f''(x)=-(1+x)^{-2}$$

$$M=\frac{2}{(1+-.1)^3}=\frac{2}{(.9)^3}\approx 2.74348422497$$

$$f'''(x)=2(1+x)^{-3}$$

$$f(x)=f(0)+\frac{f'(0)}{1}x+\frac{f''(0)}{2!}x^2+R_2(x)$$

Remainder after 2nd order term → $R_2(x)$

→

ex. 5: Find the Lagrange Error Bound when $x-\frac{x^2}{2}$ is used to approximate $\ln(1+x)$ and $|x| \le 0.1$.

Remainder Estimation Theorem

$$\left|R_n(x)\right| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$

On the interval $[-.1,.1]$, $\frac{2}{(1+x)^3}$ decreases, so its maximum value occurs at the left end-point.

$$M=\frac{2}{(1+-.1)^3}=\frac{2}{(.9)^3}\approx 2.74348422497$$

$$R_n(x) \le \frac{2.7435|.1|^3}{3!}$$

$$R_n(x) \le 0.000457$$

Lagrange Error Bound

x	$\ln(1+x)$	$x-\frac{x^2}{2}$	error
.1	.0953102	.095	.000310
–.1	–.1053605	–.105	.000361

Error is less than error bound.

π