Section 2.2 The Method of False Position

Features of BISECTION:
Plusses:
- Easy to implement
- Almost idiot proof
  - If f(x) is continuous & changes sign on [a, b], then it is GUARANTEED to converge.
- Requires ONE new function evaluation per step

Minuses:
- Slow rate of convergence; $O(1/2^n)$
- There is a theoretical bound on the error for each step, but it can overly pessimistic.
- It may take a long time to converge for a small tolerance.

The **method of false position** is also an **Enclosure or bracketing method**. For this method we will be able to remedy some of the minuses of bisection.

- Guaranteed convergence under mild conditions with linear convergence. Again we assume f(x) is continuous & changes sign on [a, b].
- We can get a fairly accurate error estimate at each step. (<u>MAIN ADVANTAGE</u>)
- The error estimate will allow the formulation of a stopping criteria that should mean we can use fewer steps.

The method of false position generates a sequence of bracketing intervals (a_n, b_n) and a sequence of approximations p_n which is in interval (a_n, b_n). The root will either be in (a_n, p_n) or (p_n, b_n). The point p_n is the x-intercept of the line connecting points

$$(a_n, f(a_n)) \text{ and } (b_n, f(b_n)).$$

The equation of the line connecting this pair of points is

$$y - f(b_n) = \frac{f(b_n) - f(a_n)}{b_n - a_n}(x - b_n).$$

Setting y = 0 and solving for x gives **x = p_n**

$$p_n = b_n - f(b_n)\frac{b_n - a_n}{f(b_n) - f(a_n)}.$$

Reciprocal of slope.

Example: Let **f(x) = x^3 + 2x^2 – 3x -1** and take [a, b] = [1, 2]. Function f is continuous on [1, 2][and f(1) = -1, f(2) = 9 so there is a root in [1, 2]. Compute the x-intercept between points (1,-1) and (2,9); Xinter = 1.1. Compute f(1.1) = -0.549. (Here **pos** and **neg** refer to the sign of the values of f at the x-value. In the tables below Xinter is a value of p_n.)

XL	XR	Xinter	f(Xinter)
1.0000**neg**	2.0000**pos**	1.1000**neg**	-0.549

$$\text{Xinter} = 2 - 9\frac{2 - 1}{9 - (-1)} = 1.1$$

So the subinterval containing the root is [1.1, 2.0]. Compute the x-intercept between points (1.1,-0.549) and (2,9); p_1 = Xinter = 1.151743638. Compute f(1.151743638) = -0.274.

$$\text{Xinter} = 2 - 9\frac{2 - 1.1}{9 - (-0.549)} = 1.151743638$$

XL	XR	Xinter	f(Xinter)
1.0000**neg**	2.0000**pos**	1.1000**neg**	-0.549
1.1000**neg**	2.0000**pos**	1.1517**neg**	-0.274

So the subinterval containing the root is [1.1517, 2.0]. Compute the x-intercept between points (1.1517,-0.274) and (2,9); p_2 = Xinter = 1.176840909. Compute f(1.176840909) = -0.1307.

XL	XR	Xinter	f(Xinter)
1.0000**neg**	2.0000**pos**	1.1000**neg**	-0.549
1.1000**neg**	2.0000**pos**	1.1517**neg**	-0.274
1.1517**neg**	2.0000**pos**	1.1768**neg**	-0.1307

We continue in this way. The Xinter values will converge to a root. Here are several more iterations. Note the decrease of the magnitude of the function values.

p_n = Xinter	f(Xinter)
1.100000000000000	-0.548999999999999
1.151743638077286	-0.274400720211667
1.176840909982786	-0.130742528809215
1.188627673293829	-0.060875863260288
1.194078911293239	-0.028040938442301

Note that the sequence of intervals is

XL	XR
1.000000000000000	2.000000000000000
1.100000000000000	2.000000000000000
1.151743638077286	2.000000000000000
1.176840909982786	2.000000000000000
1.188627673293829	2.000000000000000
1.194078911293239	2.000000000000000

+++

We see that in this example one of the original endpoints remains as an endpoint of the bracketing intervals. (We comment on this behavior later.)

What we need next is information on the **Order of Convergence**, **The Rate of Convergence**, and a **Stopping Criterion** for the algorithm. These concepts are interrelated here.

We examine the error sequence $e_n = p_n - p$ for $n \geq 0$. In this case we construct what is called the **error evolution equation**. We know that

$$p_n = b_n - f(b_n)\frac{b_n - a_n}{f(b_n) - f(a_n)}.$$

We subtract **p** from both sides to get

$$p_n - p = b_n - p - f(b_n)\frac{b_n - a_n}{f(b_n) - f(a_n)}$$

Here is where things get "tricky". What we hope to be able to do is get more information about function **f(x)** involved. In order to do this we will use <u>second degree Taylor polynomials</u>. Recall that

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2$$

We set $x_0 = p$ (so $f(x_0) = 0$) and set $x = a_n$ and then b_n to obtain two expressions:

$$f(a_n) \approx f'(p)(a_n - p) + \frac{f''(p)}{2}(a_n - p)^2$$
$$f(b_n) \approx f'(p)(b_n - p) + \frac{f''(p)}{2}(b_n - p)^2$$

We first work with the term **f(bₙ) – f(aₙ)** in the denominator;

$$f(b_n) - f(a_n) \approx f'(p)(b_n - a_n) + \frac{f''(p)}{2}\left[(b_n - p)^2 - (a_n - p)^2\right]$$
$$= (b_n - a_n)\left[f'(p) + \frac{f''(p)}{2}(b_n + a_n - 2p)\right].$$

Then using the Taylor approximation for **f(bₙ)** and the preceding approximation for **f(bₙ) – f(aₙ)** together with a bit of algebra we get that

$$p_n - p \approx (b_n - p)\left[1 - \frac{f'(p) + \frac{f''(p)}{2}(b_n - p)}{f'(p) + \frac{f''(p)}{2}(b_n + a_n - 2p)}\right]$$
$$\approx (b_n - p)(a_n - p)\frac{f''(p)}{2f'(p) + f''(p)(b_n + a_n - 2p)}$$

Now we need an important observation. We saw in the example that one end of the bracketing interval was fixed.

OBSERVATION: ==> Now, not every problem will have one endpoint fixed for all iterations; however, when using the method of false position, as the iteration proceeds, one of the endpoints will always eventually become fixed. In fact, the method of false position will always eventually settle into one of the configurations shown in Figure 2-5.
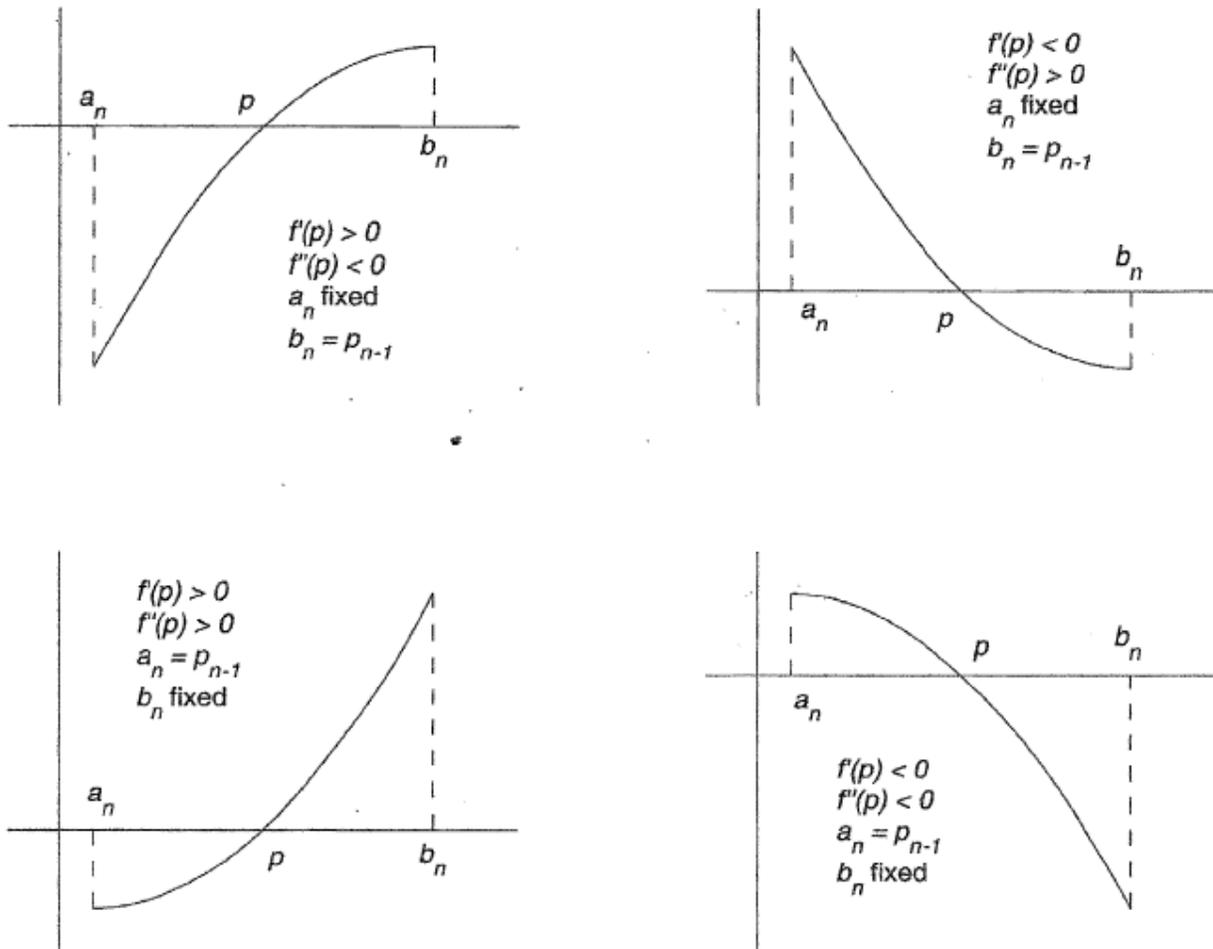
Figure 2.5 Eventual configurations for the method of false position.

Because one endpoint of the enclosing interval remains fixed the other is p_{n-1} so the following expression can be rewritten: *(just do the algebra)*

$$p_n - p \approx (b_n - p)\left[1 - \frac{f'(p) + \frac{f''(p)}{2}(b_n - p)}{f'(p) + \frac{f''(p)}{2}(b_n + a_n - 2p)}\right]$$

$$\approx (b_n - p)(a_n - p)\frac{f''(p)}{2f'(p) + f''(p)(b_n + a_n - 2p)}$$

For example if b_n is the fixed end, then $b_n - p = p_{n-1} - p = e_{n-1}$ so we have

$$p_n - p \approx (b_n - p)(a_n - p)\frac{f''(p)}{2f'(p) + f''(p)(b_n + a_n - 2p)}$$

$$e_n \approx e_{n-1}(a_n - p)\frac{f''(p)}{2f'(p) + f''(p)(e_{n-1} + a_n - p)}$$

Next we define

$$\lambda = \frac{lf''(p)}{2f'(p) + lf''(p)} \quad \text{and} \quad l = \begin{cases} a_n - p, & a_n \text{ remains fixed} \\ b_n - p, & b_n \text{ remains fixed} \end{cases}$$

So we have

$$e_n \approx \lambda e_{n-1}$$

Provided $|\lambda| < 1$, it then follows that the sequence generated by the method of false position converges, and the <u>convergence is linear with asymptotic error constant $|\lambda|$</u>.

The only question that remains is whether $|\lambda|$ really is less than 1. Remember that the method of false position eventually settles into one of the configurations shown in Figure 2-5. Let's consider the configuration in the upper left corner in more detail. Because a_n is fixed, $l = a_n - p$. Now, $a_n - p < 0$ and $f''(p) < 0$, so $(a_n - p)f''(p) > 0$. Since $f'(p)$ is also greater than zero, it follows that $2f'(p) + (a_n - p)f''(p) > (a_n - p)f''(p)$ and

$$0 < \frac{(a_n - p)f''(p)}{2f'(p) + (a_n - p)f''(p)} = \lambda < 1.$$

Hence $|\lambda| < 1$ and the method of false position converges. In a similar fashion, the remaining three configurations can be shown to lead to $|\lambda| < 1$. The details are left as an exercise.

STOPPING CRITERIA

We would like to terminate the method of false position when $|e_n|$ falls below a specified convergence tolerance, ϵ. To implement this idea, we must have a formula for estimating $|e_n|$ which involves only quantities which can be calculated during the course of the iteration. As a starting point, note that

$$\begin{aligned} e_n &= p_n - p \\ &= p_n - p_{n-1} + p_{n-1} - p \\ &= p_n - p_{n-1} + e_{n-1}. \end{aligned} \tag{5}$$

From the error evolution equation, $e_n \approx \lambda e_{n-1}$, or equivalently, $e_{n-1} \approx e_n/\lambda$. Substituting this expression into (5), solving for e_n and taking absolute values yields

$$|e_n| \approx \left| \frac{\lambda}{\lambda - 1} \right| |p_n - p_{n-1}|. \tag{6}$$

This provides an error estimate at each step. But λ involves values of the derivatives of f(x).

Next, we focus on λ. The value of λ can be estimated using terms from the sequence $\{p_n\}$ as follows. Consider the ratio

$$\frac{p_n - p_{n-1}}{p_{n-1} - p_{n-2}} = \frac{(p_n - p) - (p_{n-1} - p)}{(p_{n-1} - p) - (p_{n-2} - p)} = \frac{e_n - e_{n-1}}{e_{n-1} - e_{n-2}}.$$

Using the relations $e_n \approx \lambda e_{n-1}$ and $e_{n-2} \approx e_{n-1}/\lambda$, we find

$$\frac{p_n - p_{n-1}}{p_{n-1} - p_{n-2}} \approx \frac{(\lambda - 1)e_{n-1}}{(1 - \frac{1}{\lambda})e_{n-1}} = \lambda. \qquad \qquad (7)$$

Equations (6) and (7) together constitute a computable estimate for $|e_n|$. The accuracy of this estimate is demonstrated in Table 2-3. Consequently, an appropriate stopping condition for the method of false position is to terminate the iteration when

$$\left| \frac{\lambda}{\lambda - 1} \right| |p_n - p_{n-1}| < \epsilon,$$

where λ is obtained from (7).

Note: this "estimator" is used after we have determined 3 x-intercepts using the false position strategy.

Example: METHOD of FALSE POSITION

For f = x^3+2*x^2-3*x-1 with initial interval [1,2] where the tolerance is 0.005 and maxit = 8.

```
   Approximations  and  Function values
 1.000000000000000  -1.000000000000000
 2.000000000000000   9.000000000000000
 1.100000000000000  -0.548999999999999
 1.151743638077286  -0.274400720211667
 1.176840909982786  -0.130742528809215
 1.188627673293829  -0.060875863260288
 1.194078911293239  -0.028040938442301
 1.196582088205248  -0.012852240234538
```

```
Approximations  and  Error Estimates
1.000000000000000          0       ← initial data
2.000000000000000          0       ← initial data
1.100000000000000          0       ← first x-intercept = p₁
1.151743638077286          0       ← second x-intercept = p₂
1.176840909982786   0.023638234686033   ← first error estimate
1.188627673293829   0.010437451609353
1.194078911293239   0.004690375977452   ← error less than tolerance
1.196582088205248
```

Another Example: Estimate a root of f(x) = cos(x) − x via false position.

Let $f(x) = \cos x - x$. For the first iteration, we have $(a_1, b_1) = (0, 1)$ and we know that $f(a_1) = 1 > 0$ and that $f(b_1) = \cos 1 - 1 \approx -0.460 < 0$. Our first approximation to the location of the root is

$$p_1 = b_1 - f(b_1)\frac{b_1 - a_1}{f(b_1) - f(a_1)} = 0.685073357.$$

To determine whether the root is contained on (a_1, p_1) or on (p_1, b_1), we calculate $f(p_1) \approx 0.0893 > 0$. Since $f(a_1)$ and $f(p_1)$ are of the same sign, the Intermediate Value Theorem tells us that root is between p_1 and b_1. For the next iteration, we therefore take $(a_2, b_2) = (p_1, b_1) = (0.685073357, 1)$. Our second approximation to the location of the root is

$$p_2 = b_2 - f(b_2)\frac{b_2 - a_2}{f(b_2) - f(a_2)} = 0.736298998.$$

Note that $f(p_2) \approx 0.00466 > 0$, which is of the same sign as $f(a_2)$. Hence, the Intermediate Value Theorem tells us the root is between p_2 and b_2, so we take $(a_3, b_3) = (p_2, b_2) = (0.736298998, 1)$. In the third iteration, we calculate

$$p_3 = b_3 - f(b_3)\frac{b_3 - a_3}{f(b_3) - f(a_3)} = 0.738945356$$

and $f(p_3) \approx 2.339 \times 10^{-4} < 0$. Hence, we find that $f(a_3)$ and $f(p_3)$ are of the same sign, which implies that the root lies somewhere between p_3 and b_3. For the fourth iteration, we will therefore take $(a_4, b_4) = (p_3, b_3) = (0.738945356, 1)$.