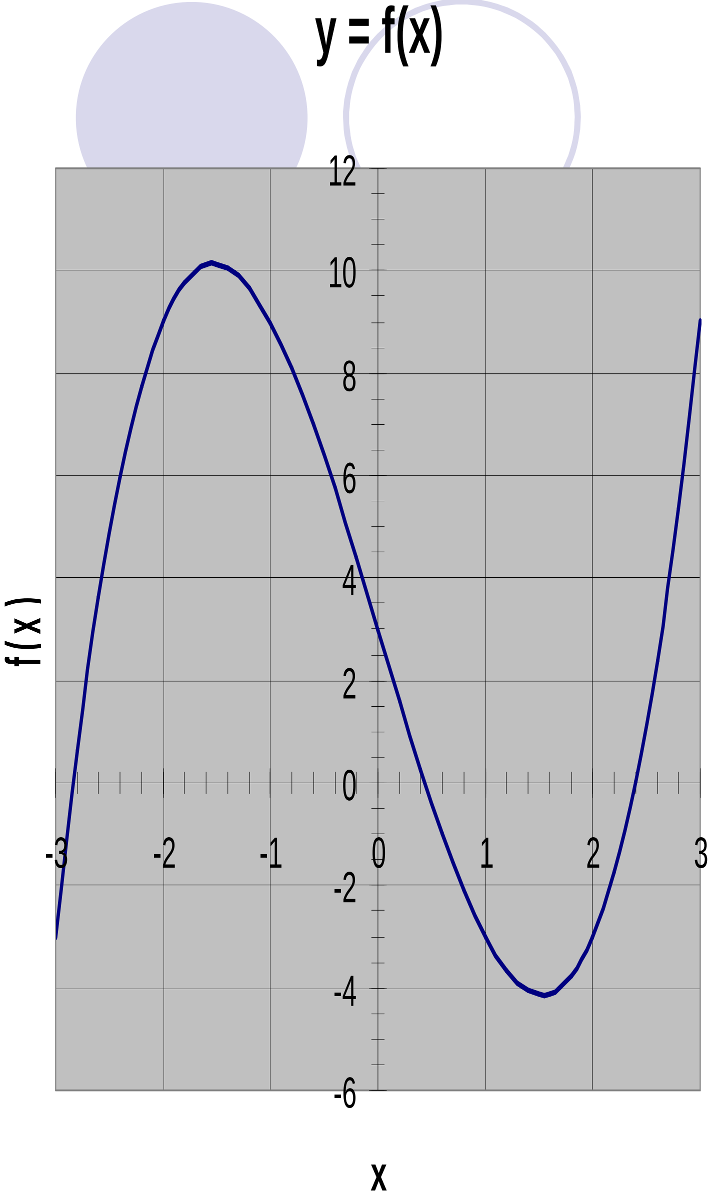
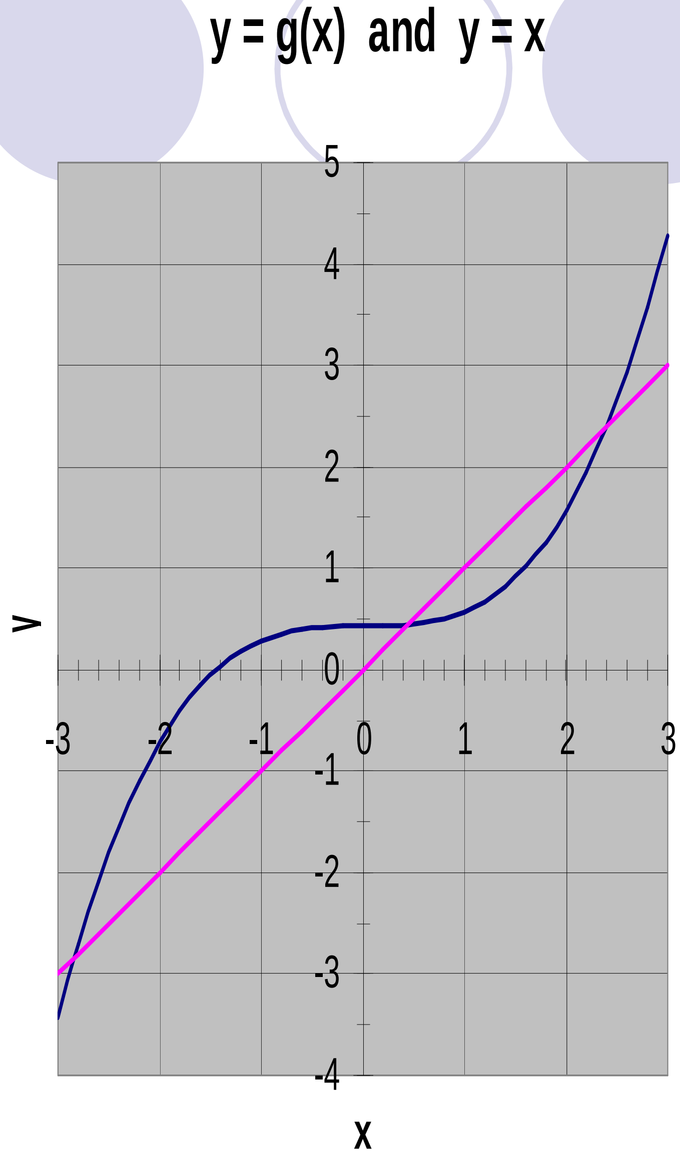
Numerical Methods

Finding Roots

Fixed Point Iteration

- Rewrite $f(x) = 0$ to $x = g(x)$
- To solve $x = g(x)$, we iteratively calculate
  - $x_{i+1} = g(x_i)$
- The problem is how to choose $g(x)$ so as to ensure convergence

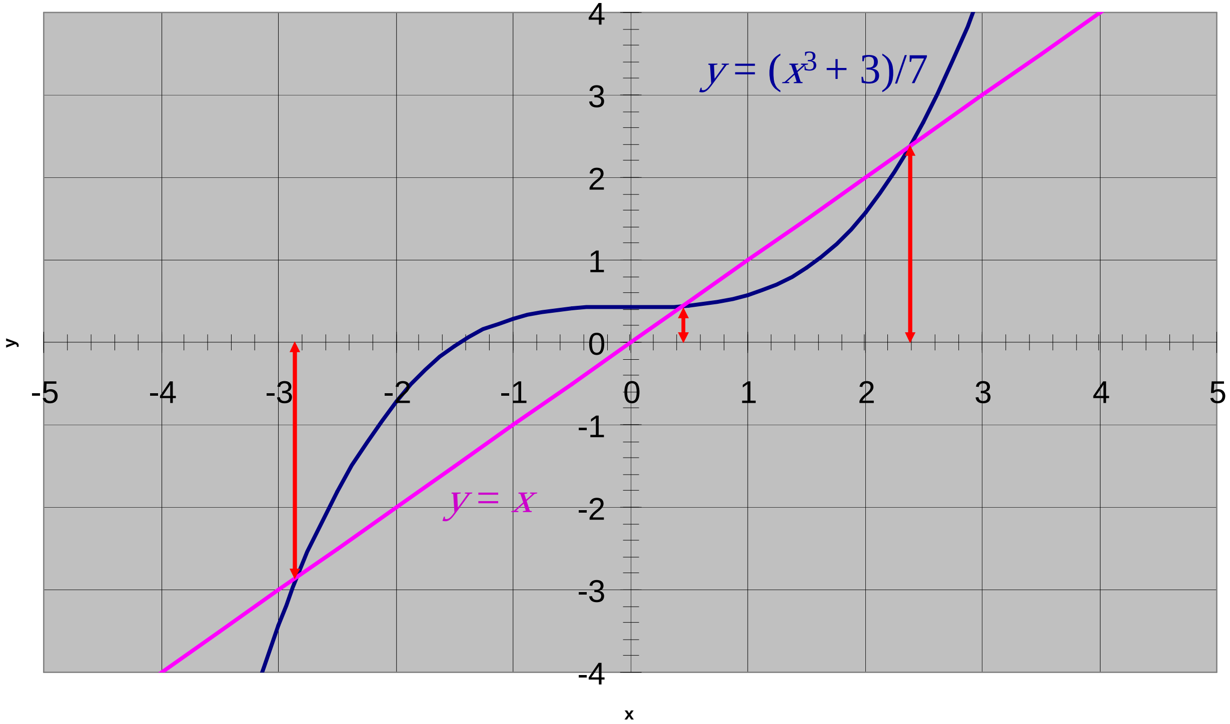

y = f(x)
f (x)
x
12
10
8
6
4
2
0
-2
-4
-6
-3
-2
-1
0
1
2
3


y = g(x) and y = x
y
x
5
4
3
2
1
0
-1
-2
-3
-4
-3
-2
-1
0
1
2
3

Fixed Point Iteration

The equation f(x) = 0, where f(x) = $x^3 - 7x + 3$, may be re-arranged to give $x = (x^3 + 3)/7$.

Intersection of the graphs of $y = x$ and $y = (x^3 + 3)/7$ represent roots of the original equation $x^3 - 7x + 3 = 0$.



Fixed Point Iteration

The rearrangement $x = (x^3 + 3)/7$ leads to the iteration

$$x_{n+1} = \frac{x_n^{\ 3} + 3}{7}, \quad n = 0, 1, 2, 3, \ldots$$

To find the middle root α, let initial approximation $x_0 = 2$.

$$x_1 = \frac{x_0^{\ 3} + 3}{7} = \frac{2^3 + 3}{7} = 1.57143$$

$$x_2 = \frac{x_1^{\ 3} + 3}{7} = \frac{1.57143^3 + 3}{7} = 0.98292$$

$$x_3 = \frac{x_2^{\ 3} + 3}{7} = \frac{0.98292^3 + 3}{7} = 0.56423$$

$$x_4 = \frac{x_3^{\ 3} + 3}{7} = \frac{0.56423^3 + 3}{7} = 0.45423$$

etc.

The iteration slowly converges to give α = **0.441** (to 3 s.f.)

Fixed-point Iteration Example(1)

To solve

$$x - x^{1/3} - 2 = 0$$

rewrite as

$$x_{\text{new}} = g_1(x_{\text{old}}) = x_{\text{old}}^{1/3} + 2$$

or

$$x_{\text{new}} = g_2(x_{\text{old}}) = \left(x_{\text{old}} - 2\right)^3$$

or

$$x_{\text{new}} = g_3(x_{\text{old}}) = \frac{6 + 2x_{\text{old}}^{1/3}}{3 - x_{\text{old}}^{2/3}}$$

Fixed-point Iteration Example(2)

$$g_1(x) = x^{1/3} + 2$$

$$g_2(x) = (x - 2)^3$$

$$g_3(x) = \frac{6 + 2x^{1/3}}{3 - x^{2/3}}$$

k	$g_1(x_{k-1})$	$g_2(x_{k-1})$	$g_3(x_{k-1})$
0	3	3	3
1	3.4422495703	1	3.5266442931
2	3.5098974493	-1	3.5213801474
3	3.5197243050	-27	3.5213797068
4	3.5211412691	-24389	3.5213797068
5	3.5213453678	-1.451×10^{13}	3.5213797068
6	3.5213747615	-3.055×10^{39}	3.5213797068
7	3.5213789946	-2.852×10^{118}	3.5213797068
8	3.5213796042	∞	3.5213797068
9	3.5213796920	∞	3.5213797068

Fixed point theorem

A *continuous* f is *contractive* if there is an $L < 1$ such that

$$|f(x) - f(y)| \leq L|x - y|$$

for all x, y in the domain of f.

Note: If $|f'(x)| < 1$ for all $x \in [a, b]$, then f is contractive in $[a, b]$

$g : [a, b] \to \mathcal{R}$ has a unique fixed point if:

- $g : [a, b] \to [a, b]$ (assures existence)
- g is contractive (assures uniqueness)

Fixed Point Iteration

A fixed point iteration has the form $p_{k+1} = g(p_k)$

If g is continuous and $\lim_{n\to\infty} g(p_n) = P$, then P is a fixed point of g.

If g and g' are continuous in $[a,b]$,
$g(x) \in [a,b]$ for all $x \in [a,b]$ and
$p_0 \in [a,b]$, then

- $|g'(P)| \leq K < 1 \Rightarrow \{p_n\} \longrightarrow P$
- $|g'(P)| > 1 \Rightarrow \{p_n\}$ will not converge to P

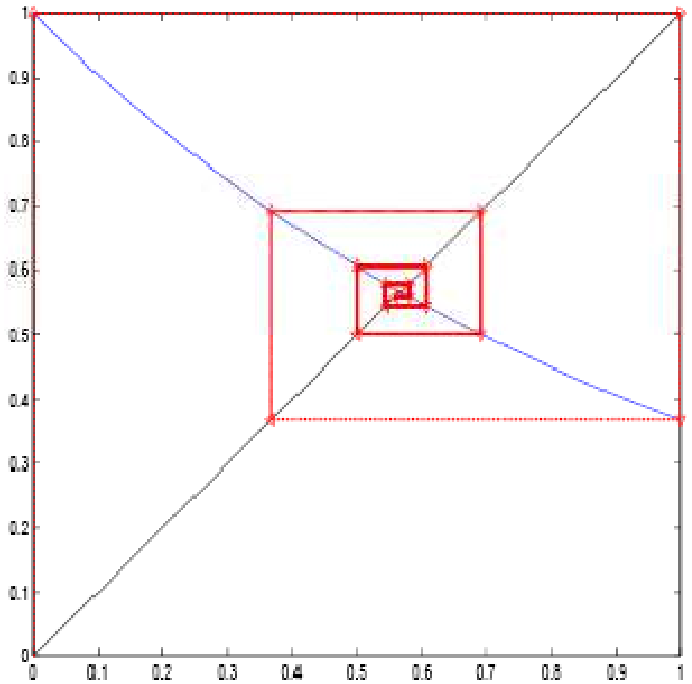
Convergent Iteration

$$x - e^{-x} = 0 \quad \Rightarrow \quad x_{k+1} = e^{-x_k},\ x_0 = 0$$

$$g(x) = e^{-x},$$

$$g : [0,1] \to [0,1]$$

$|g'(x)| = e^{-x} < 1$ for $x > 0$



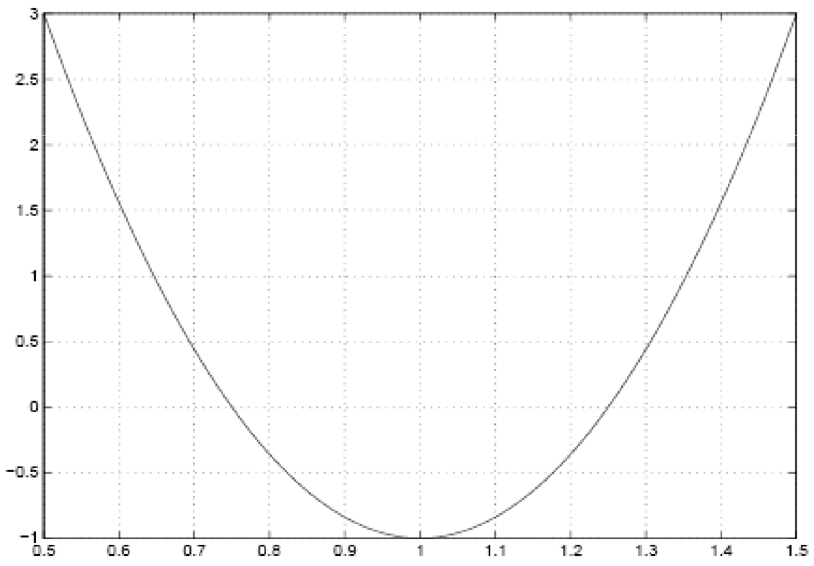
Example: Fixed Point Iteration

Find the largest root of $16x^2 - 32x + 15 = 0$ by fixed point iteration.

Some possibilities:

1. $x = 16x^2 - 31x + 15$
2. $x = \sqrt{32x - 15}/4$
3. $x = x^2/2 + 15/32$
4. $x = -15/16(x - 2)$
5. $x = 2 - 15/16x$

Ex., plot of $f(x) = 16x^2 - 32x + 15$



Take $x \in [1.2, 1.3]$

Ex., check hypothesis

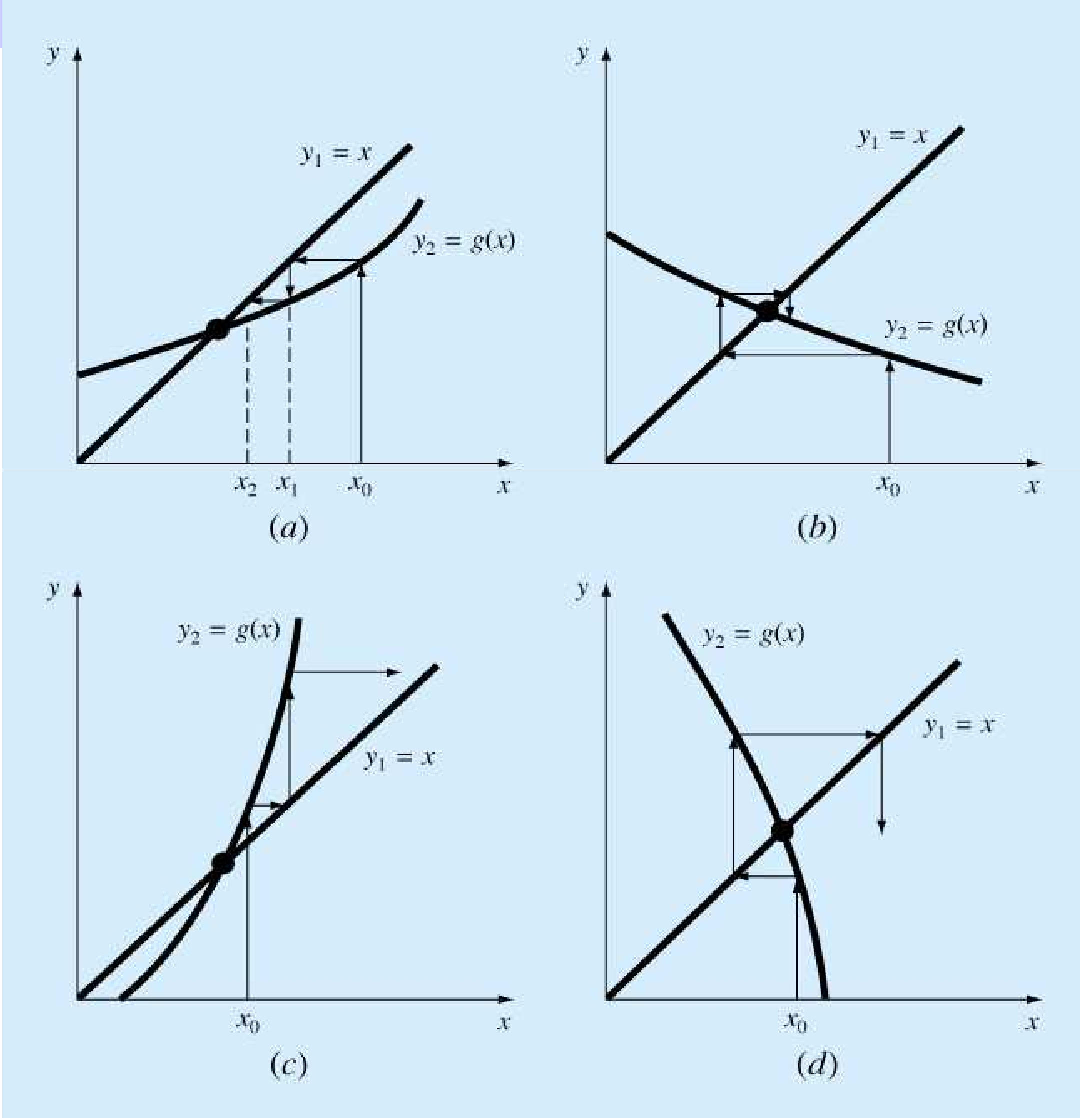
1. $g(x) = 16x^2 - 31x + 15$
   $|g'(x)| = |32x - 31| > 1$
2. $g(x) = \sqrt{32x - 15}/4$
   $|g'(x)| = 4/\sqrt{32x - 15} < 1$ and $g : [1.2, 1.3] \to [1.2, 1.3]$
3. $g(x) = x^2/2 + 15/32,$
   $|g'(x)| = |x| > 1$
4. $g(x) = -15/16(x - 2),$
   $g : [1.2, 1.3] \to [1.17, 1.34]$
5. $g(x) = 2 - 15/16x$
   $g : [1.2, 1.3] \to [1.21, 1.28]$ and $|g'(x)| = 15/16x^2 < 1$

Ex., formulations #2 and 5

$g(x) = \sqrt{32x - 15}/4$	$g(x) = 2 - 15/16x$
$x_0 = 1.2000$	$x_0 = 1.2000$
$x_1 = 1.2093$	$x_1 = 1.2188$
$x_2 = 1.2170$	$x_2 = 1.2308$
$\vdots$	$\vdots$
$x_{12} = 1.2463$	$x_{12} = 1.2499$
$x_{13} = 1.2470$	$x_{13} = 1.2499$
$\vdots$	
$x_{27} = 1.2499$	
$x_{28} = 1.2499$	

Convergence of Fixed Point Iteration

- $|g'(x)| < 1$: (a), (b)
- $|g'(x)| > 1$: (c), (d)



Example

- Find a root of $x^3 - x^2 - 1 = 0$ with $x_0 = 1.5$

	$x - 1 - x^{-2} = 0$	$x^3 = 1 + x^2$	$x^2(x - 1) = 1$
$g(x)$	$1 + x^{-2}$	$(1 + x^2)^{1/3}$	$(x - 1)^{-1/2}$
$g'(x)$	$-2x^{-3}$	$2x / [3(1+x^2)^{2/3}]$	$-1 / [2(x-1)^{3/2}]$
Comment	$\lvert g'(x)\rvert < 1$ for $x > 1.3$	$\lvert g'(x)\rvert < 1$ for $1 \leq x \leq 2$	$\lvert g'(x)\rvert > 1$ for $x < 1.6$

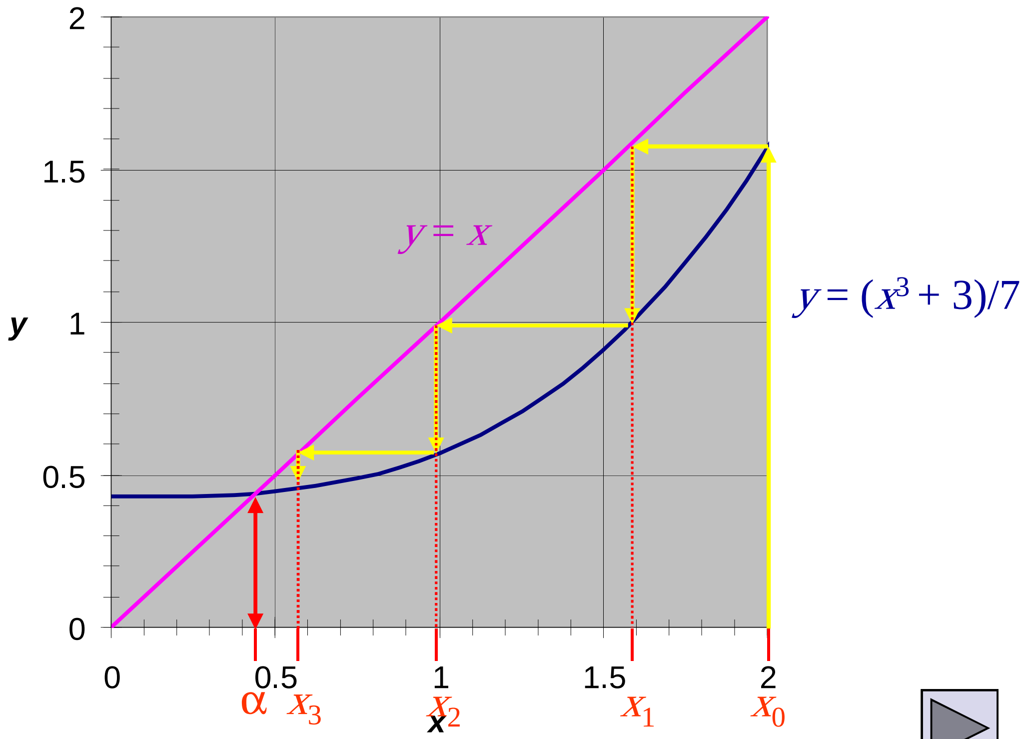
Fixed Point Iteration

The rearrangement $x = (x^3 + 3)/7$ leads to the iteration

$$x_{n+1} = \frac{x_n^{\,3} + 3}{7}, \quad n = 0, 1, 2, 3, \ldots$$

For $x_0 = 2$ the iteration will converge on the middle root α, since g'(α) < 1.

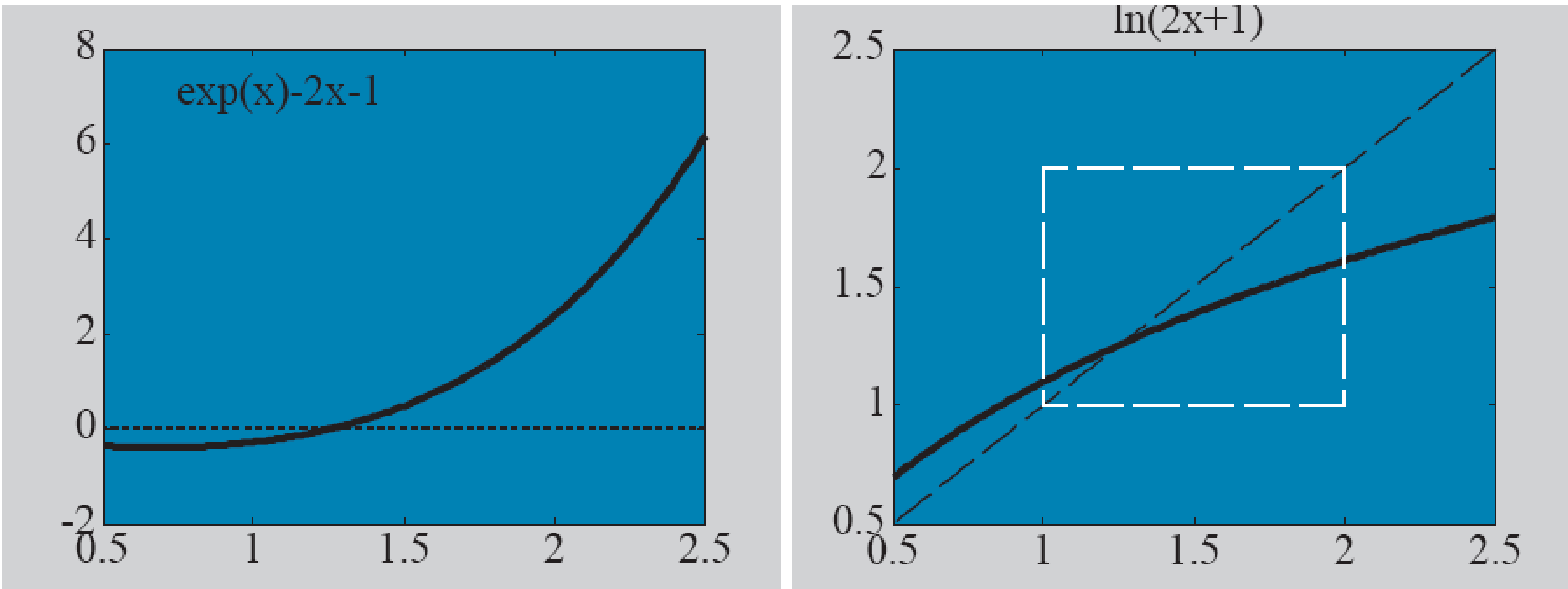
n	x
0	2
1	1.57143
2	0.98292
3	0.56423
4	0.45423
5	0.44196
6	0.4409
7	0.44082
8	0.44081



α = **0.441** (to 3 s.f.)

1.5 Example

Root-finding problem: $0 = \exp(x) - 2x - 1$
Fixed-point problem: $x = \ln(2x + 1)$



and $g'(x) = 2/(2x+1)$. $g'(x)$ is a decreasing function. Thus $\max_{\xi \in [1,2]} |g'(x)| = |g'(1)| = 2/3$

We set $L = 2/3$.

1.13 Error bounds

We get

$$\begin{aligned} |x_k - \xi| &= |g(x_{k-1}) - g(\xi)| \\ &\leq L|x_{k-1} - \xi| \\ &\leq L(|x_{k-1} - x_k| + |x_k - \xi|) \end{aligned}$$

and consequently:

$$|x_k - \xi| \leq \frac{L}{1-L}|x_k - x_{k-1}|$$

This is called an a-posteriori estimate.

1.14 Error bounds (Conts.)

Analogously we can derive an a-priori bound

$$|x_k - \xi| \leq \frac{L^k}{1-L}|x_1 - x_0|$$

How many iterations do we need in our example for an accuracy of $|x_k - \xi| \leq 10^{-8}$?

1.15 Rate of Convergence

Definition. [1.4] *Assume* $\lim_{k\to\infty} x_k = \xi$.

- *Linear convergence, if*

$$|x_k - \xi| < \epsilon_k \quad \textit{and} \quad \lim_{k\to\infty} \frac{\epsilon_{k+1}}{\epsilon_k} = \mu \quad \textit{with} \quad \mu \in (0,1)$$

- *Superlinear convergence if* $\mu = 0$.
- *Sublinear convergence if* $\mu = 1$, *e.g.* $\epsilon_k = \frac{1}{k+1}$
- *Asymptotic rate of convergence* $\rho = -\log_{10} \mu$

ρ large $\Rightarrow$ fast (linear) convergence

ρ small $\Rightarrow$ slow (linear) convergence