المحاضرة: ٧

نظري

السنة: الرابعة

التاريخ: /٣/٢٠٢١

القسم: رياضيات تطبيقية

المادة: الحلول العددية للمعادلات التفاضلية

عددالصفحات:

بعض الطرائق العددية المستخدمة في الفروق المنتهية :

① طريقة Lax-Frredrichs : (L FM).

لتكن لدينا المعادلة التفاضلية : $u_t + a u_x = 0$

نطبق الفروق التقدمية بالنسبة للزمن والمركزية بالنسبة للموضع :

$$\frac{U_i^{n+1}-U_i^n}{\Delta t} = -a\frac{[U_{i+1}^n - U_{i-1}^n]}{2h}$$

$$U_i^{n+1} - U_i^n = \frac{-a\Delta t}{2h}[U_{i+1}^n - U_{i-1}^n]$$

نضع $U_i^n = \frac{U_{i-1}^n + U_{i+1}^n}{2}$

$$U_i^{n+1} = \frac{1}{2}[U_{i-1}^n + U_{i+1}^n] + a\left(\frac{\Delta t}{2h}\right)[U_{i+1}^n - U_{i-1}^n]$$

نعوض

نحصل على معادلة الفروق المنتهية .

② طريقة Up Wind :

نأخذ الطريقة التراجعية للموضع و التقدمية للزمن :

فنحصل على المعادلة :

$$U_i^{n+1} = U_i^n - a\frac{\Delta t}{h}(U_i^n - U_{i-1}^n)$$

٣) طريقة (Beam_warming):

$$U_i^{j+1} = U_i^j - a\,\frac{\Delta t}{2h}\left(3U_i^j - 4U_{i-1}^j + U_{i-2}^j\right) + \frac{(a\Delta t)^2}{2(h)^2}\left(U_i^j - 2U_{i-1}^j + U_{i-2}^j\right)$$

3.6.2 Summary

In what follow I summarize the most salient aspects of the different finite-difference operators discussed so far and report, for each of them, the truncation error ϵ_T, the amplification factor $|\lambda|^2$ and the finite-difference representation of the advection equation 3.2.

$u_t = cu_x$

Method	ϵ_T	$\|\lambda\|^2$ for $(k\Delta x \ll 1)$	finite-difference form
Upwind	$\mathcal{O}(\Delta t, \Delta x)$	$1 - 2\|\alpha\|(1-\|\alpha\|)\cos(k\Delta x)$	$u_j^{n+1} = u_j^n \mp \alpha(u_{j\pm1}^n - u_j^n)$
FTCS	$\mathcal{O}(\Delta t, \Delta x^2)$	$1 + \sin^2(k\Delta x)\alpha^2$	$u_j^{n+1} = u_j^n - \alpha(u_{j+1}^n - u_{j-1}^n)$
Lax Friedrichs	$\mathcal{O}(\Delta t, \Delta x^2)$	$1 - \sin^2(k\Delta x)(1-\alpha^2)$	$u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - \alpha(u_{j+1}^n - u_{j-1}^n)$
Lepafrog	$\mathcal{O}(\Delta t^2, \Delta x^2)$	1	$u_j^{n+1} = u_j^{n-1} - \alpha(u_{j+1}^n - u_{j-1}^n)$
Lax Wendroff	$\mathcal{O}(\Delta t^2, \Delta x^2)$	$1 - \alpha^2(1-\alpha^2)\sin^2(k\Delta x)$	$u_j^{n+1} = u_j^n - \frac{1}{2}\alpha(u_{j+1}^n - u_{j-1}^n) - \frac{1}{2}\alpha^2(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$

Table 3.: Schematic summary of the finite-difference operators discussed so far.

(3)

Truncation Error of θ-method : خطأ الاقتطاع بطريقة تيتا

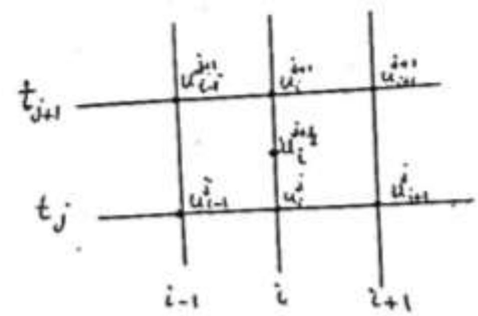
لتكن لدينا المعادلة من الشكل

$$\underbrace{u_t}_{\text{تقدمية}} = \underbrace{u_{xx}}_{\text{مركزية}}$$

$$(\star) \quad \frac{u_i^{j+1} - u_i^j}{k} = \theta \underbrace{\left[\frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{h^2}\right]}_{A} + (1-\theta)\underbrace{\left[\frac{u_{i+1}^{j} - 2u_i^{j} + u_{i-1}^{j}}{h^2}\right]}_{B}$$

نوسّع الحدود في خطأ الاقتطاع ليشمل $u(x_i, t_{j+\frac{1}{2}})$

[هذا الشكل يوضح توضّع النقطة $u_i^{j+\frac{1}{2}}$: $u(x_i, t_{j+\frac{1}{2}})$]



بالنشر حسب تايلور نجد :

$$u_i^{j+1} = \left[u + (\tfrac{1}{2}k) u_t + \tfrac{1}{2}(\tfrac{1}{2}k)^2 u_{tt} + \tfrac{1}{3!}(\tfrac{1}{2}k)^3 u_{ttt} + \cdots\right]_i^{j+\frac{1}{2}}$$

$$u_i^{j} = \left[u - (\tfrac{1}{2}k) u_t + \tfrac{1}{2}(\tfrac{1}{2}k)^2 u_{tt} - \tfrac{1}{3!}(\tfrac{1}{2}k)^3 u_{ttt} + \cdots\right]_i^{j+\frac{1}{2}}$$

$$\Longrightarrow \quad \frac{u_i^{j+1} - u_i^j}{k} = \left[u_t + \tfrac{1}{24}k^2 u_{ttt} + \cdots\right]_i^{j+\frac{1}{2}}$$

وهذا هو الطرف الأيسر من العلاقة ★ :

وأيضاً نحن نعلم :

$$A = \frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{h^2} = \left[u_{xx} + \frac{1}{12} h^2 u_{xxxx} + \cdots \right]_i^{j+1}.$$

$$B = \frac{u_{i+1}^{j} - 2u_i^{j} + u_{i-1}^{j}}{h^2} = \left[u_{xx} + \frac{1}{12} h^2 u_{xxxx} + \cdots \right]_i^{j}$$

لنوسّع الحدود من الطرف الأيمن لتشمل النقاط $u_i^{j+\frac{1}{2}}$

وذلك بالنشر حسب تايلور لكل حد من حدود الطرف الأيمن (u_{xx} و u_{xxxx})

$$\Rightarrow A = \left[u_{xx} + \left(\frac{k}{2}\right) u_{xxt} + \frac{1}{2}\left(\frac{k}{2}\right)^2 u_{xxtt} + \cdots \right]_i^{j+\frac{1}{2}}$$

$$+ \frac{1}{12} h^2 \left[u_{xxxx} + \left(\frac{k}{2}\right) u_{xxxxt} + \cdots \right]_i^{j+\frac{1}{2}}$$

$$\overset{\text{وأيضاً}}{\Longrightarrow} B = \left[u_{xx} - \left(\frac{k}{2}\right) u_{xxt} + \frac{1}{2}\left(\frac{k}{2}\right)^2 u_{xxtt} + \cdots \right]_i^{j+\frac{1}{2}}$$

$$+ \frac{1}{12} h^2 \left[u_{xxxx} - \left(\frac{k}{2}\right) u_{xxxxt} + \cdots \right]_i^{j+\frac{1}{2}}$$

$$\theta A + (1-\theta) B = [\theta + 1 - \theta]\left[u_{xx} + \frac{k^2}{8} u_{xxtt} + \frac{1}{12} h^2 u_{xxxx} + \cdots \right]_i^{j+\frac{1}{2}}$$

$$+ [\theta - 1 + \theta] \left[\frac{k}{2} u_{xxt} + \frac{h^2}{12}\frac{k}{2} u_{xxxxt} + \cdots \right]_i^{j+\frac{1}{2}}$$

$$= \Big[u_{xx} + \frac{k^2}{8} u_{xxtt} + \frac{1}{12} h^2 u_{xxxx} + \cdots$$

$$+ (2\theta - 1)\frac{k}{2}\left(u_{xxt} + \frac{1}{12} h^2 u_{xxxxt} + \cdots \right) \Big]_i^{j+\frac{1}{2}}$$

وهذا هو الطرف الأيمن من العلاقة ★

وبما أن $u_{xx} - u_t = 0$ ⇐ نجد أن :

$$T_i^{j+\frac{1}{2}} = \frac{1}{2}(1-2\theta)\, K u_{xxt} - \frac{1}{12} h^2\, u_{xxxx} + K^2\left[\frac{1}{24} u_{ttt} - \frac{1}{8} u_{xxtt}\right] + \cdots$$

وبما أن $u_{xx} = u_t$, $u_{xxxx} = u_{tt}$, ... تصبح العلاقة :

$$T_i^{j+\frac{1}{2}} = \frac{1}{2}(1-2\theta) K\, u_{xxxx} - \frac{1}{12} h^2\, u_{xxxx} - \frac{K^2}{12}\, u_{xxxxxx} + \cdots$$

في الحالة الخاصة $\theta = \frac{1}{2}$ يصبح الحد الأول صفراً

وتكون الخطأ رتبة ثانية بالنسبة للموضع والزمن $O(h^2, \Delta t^2)$

و هذه تدعى طريقة كرانك نيكلسون .

The Leapfrog Method

الفكرة في هذه الطريقة أننا نزيد من الدقة (بالنسبة للزمن t) باستخدام الفروق المركزية .

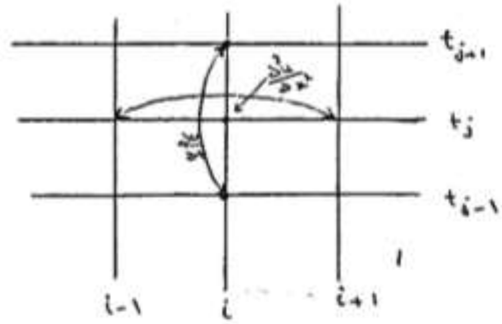
$$\underset{\swarrow\ \text{مركزية}}{\frac{\partial u}{\partial t}} = \underset{\text{مركزية}\ \searrow}{\frac{\partial^2 u}{\partial x^2}}$$

$$\frac{u_i^{j+1} - u_i^{j-1}}{2K} = \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2}$$

تصبح المعادلة :

$$(1)\quad u_i^{j+1} = u_i^{j-1} + 2r\left(u_{i+1}^j - 2u_i^j + u_{i-1}^j\right) \qquad r = \frac{K}{h^2}$$

وبسهولة يمكن أن نجد : $T_i^j \sim O(K^2, h^2)$



إذا نظرنا إلى صيغة فورييه (نيومان) يعني

$$u_i^j \sim \lambda^j e^{ikih}$$

تصبح المعادلة ① ⟸

$$\lambda = \frac{1}{\lambda} + 2r\left(e^{ikh} - 2 + e^{-ikh}\right)$$

$$\lambda = \frac{1}{\lambda} + 2r\left(2\cos(kh) - 2\right) \quad : \left[\cos x = \frac{e^{ix} + e^{-ix}}{2}\right]$$

$$\lambda = \frac{1}{\lambda} - 8r \underbrace{\sin^2 \frac{kh}{2}}_{p} \quad , \quad \left[\sin^2 \frac{x}{2} = \frac{1 - \cos x}{2}\right]$$

لنسمي $\left(p = \sin^2 \frac{kh}{2}\right)$

فنجد : $0 \leq p \leq 1$

$$\Longrightarrow \lambda^2 + 8rp\lambda - 1 = 0$$

بحل المعادلة :

$$\Delta = (8rp)^2 + 4$$
$$= 4(16r^2p^2 + 1)$$

$$\Longrightarrow \lambda_{1,2} = -4rp \mp \sqrt{16r^2p^2 + 1}$$

$$\left.\begin{aligned} \lambda_1 &= \sqrt{16r^2p^2+1} - 4rp < 1 \\ \lambda_2 &= -\sqrt{16r^2p^2+1} - 4rp < -1 \end{aligned}\right\} \Longrightarrow |\lambda_2| > 1$$

إذاً غير مستقرة.

« المطالبة »

الاستقرار Stability

تكون طريقة الفروق المركزية مستقرة : إذا $\|u^j\|_{\ell_2} \leq \|u^0\|_{\ell_2}$

عندما : $\|u^j\|_{\ell_2} = \left[h \sum_{-\infty}^{\infty} |u_i^j|^2\right]^{\frac{1}{2}}$

نحن نستخدم تحويلات فوريه (متقطع)

$$\hat{u}(k) = h \sum_{-\infty}^{+\infty} u_i e^{-ikrh} \qquad : k \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]$$

$$\Leftarrow \quad u_i = \frac{1}{2\pi} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \hat{u}(k) e^{ikrh} \, dk$$

ومطابقة بارسيفال :

$$\|u\|_{\ell_2} = \frac{1}{\sqrt{2\pi}} \|\hat{u}\|_{L_2}$$

عندما :

$$\|\hat{u}\|_{L_2} = \left(\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} |\hat{u}(k)|^2 dk\right)^{\frac{1}{2}}$$

وإذا أخذنا المعادلة الظاهرية :

$$u_i^{j+1} = u_i^j + r(u_{i+1}^j - 2u_i^j + u_{i-1}^j)$$

وإذا أخذنا تحويل فوريه :

$$h \sum_{i=-\infty}^{\infty} u_i^{j+1} e^{iki h} = h \sum_{-\infty}^{+\infty} u_i^j e^{ikih} + r\left(h \sum_{-\infty}^{\infty} e^{ikih} (u_{i+1}^j - 2u_i^j + u_{i-1}^j)\right)$$

أي :

$$\hat{u}^{j+1}(k) = \hat{u}^j(k) + r(e^{ikh} - 2 + e^{-ikh}) \hat{u}^j(k)$$

$$\Leftarrow \quad \hat{u}^{j+1}(k) = \hat{u}^j(k) \left[1 - 4r \sin^2\left(\frac{kh}{2}\right)\right]$$

لنفرض

$$\lambda(k) = 1 - 4r \sin^2\left(\frac{kh}{2}\right)$$

فيصبح لدينا

$$\hat{u}^{j+1} = \hat{u}^j(k) \cdot \lambda(k)$$

وباستخدام متطابقة بارسيفال :

$$\|U^{j+1}\|_{\ell_2} = \frac{1}{\sqrt{2\pi}} \|\hat{U}^{j+1}\|_{L_2}$$

$$= \frac{1}{\sqrt{2\pi}} \|\lambda \hat{u}^{j}\|_{L_2}$$

$$\|U^{j+1}\|_{\ell_2} \leq \max_{K} |\lambda| \, \|U^{j}\|_{\ell_2}$$

لذلك إجمالاً : $\max\limits_{K} |\lambda(K)| \leq 1$ شرط كاف للاستقرار عملياً

ونتيجة لذلك : $-1 \leq 1 - 4r\sin^2\frac{Kh}{2} \leq 1$

تكون مستقرة في حالة $\left(r \leq \frac{1}{2} \right)$ ...

* إذا طبقنا نفس الطريقة على المعادلة الضمنية (على الشكل الضمني)

نحصل على : $\hat{U}^{j+1}(K) = \lambda(K)\, \hat{U}^{j}(K)$

ولكن تكون : $\lambda(K) = \dfrac{1}{1 + 4r\sin^2\frac{Kh}{2}}$

والاستقرار يتحقق أياً كانت r (من أجل جميع قيم r) .

Example 52. *Forward Euler method for $u_t - u_{xx} = 0$*

We saw that $T(x,t) = O(k + h^2)$, so if $k, h \to 0$ then $T \to 0$. So forward Euler is unconditionally consistent with $u_t - u_{xx} = 0$.

Convergence: Now suppose we carry out a sequence of calculations with the same initial data, but with successive mesh refinement so that $k, h \to 0$. We call the method convergent if, for any fixed point (x_*, t_*),

متماسكة غير متقاربة
والسبب شروط البدء

$$x_j \to x_*, \quad t_n \to t_* \quad \Longrightarrow \quad U_j^n \to u(x_*, t_*).$$

Warning! consistency $\neq$ convergence. To get U_j^n we apply the formula successively, so the individual local truncation errors could add up.

Example 53. *Forward differences for $u_t + u_x = 0$, in $t > 0$, $x \in \mathbb{R}$, with initial data*

$$u(x,0) = u_0(x) = \begin{cases} 1 & x < 0, \\ 0 & x \geq 0. \end{cases}$$

Note that the exact solution is $u(x,t) = u_0(x - t)$ (Figure 43). Try forward differences in both derivatives:

$$\frac{U_j^{n+1} - U_j^n}{k} + \frac{U_{j+1}^n - U_j^n}{h} = 0,$$
$$\Longrightarrow \quad U_j^{n+1} = (1 + \lambda) U_j^n - \lambda U_{j+1}^n,$$

where $\lambda = k/h$ (called the Peclet number).

Check the scheme for consistency by Taylor expanding:

$$\frac{u(x_j, t_n + k) - u(x_j, t_n)}{k} = u_t + \tfrac{k}{2} u_{tt} + O(k^2),$$
$$\frac{u(x_j + h, t_n) - u(x_j, t_n)}{h} = u_x + \tfrac{h}{2} u_{xx} + O(h^2).$$

So the local truncation error is

$$T(x,t) = O(k + h) \quad \to 0 \quad \text{as } k, h \to 0,$$

so the scheme is consistent with the PDE.

Since we know the exact solution we can show that the scheme does not converge by considering $x_* = 0$, $t_* > 0$. Initially, $U_j^0 = u_0(jh) = 0$ for all $j \geq 0$. But induction on n with $U_0^{n+1} = (1 + \lambda) U_0^n - \lambda U_1^n$ for $j \geq 0$ shows that $U_0^{n+1} = 0$ for all $n \geq 0$. Since

$$u(0, (n+1)k) = u_0(-(n+1)k) = 1,$$

we see that $U_0 \not\to u_0(0, nk)$ as $k \to 0$ with $t_n = nk$ fixed. So although the method is consistent it does not converge.

We saw in MATLAB an example where convergence of Euler's method depended on the relative step sizes k and h.

Example 54. *Prove that the forward Euler method for $u_t - u_{xx} = 0$ is convergent if $\mu = k/h^2 \leq \frac{1}{2}$.*

Let $e_j^n = U_j^n - u(x_j, t_n)$, sometimes called the "global truncation error". By definition,

$$T(x_j, t_n) = \frac{u(x_j, t_{n+1}) - u(x_j, t_n)}{k} - \frac{u(x_{j+1}, t_n) - 2u(x_j, t_n) + u(x_{j-1}, t_n)}{h^2},$$

$$\implies u(x_j, t_{n+1}) = kT(x_j, t_n) + u(x_j, t_n) + \mu\Big(u(x_{j+1}, t_n) - 2u(x_j, t_n) + u(x_{j-1}, t_n)\Big).$$

So

$$\begin{aligned}
|e_j^{n+1}| &= \left|U_j^{n+1} - u(x_j, t_{n+1})\right| \\
&= \Big|U_j^n + \mu(U_{j+1}^n - 2U_j^n + U_{j-1}^n) - kT(x_j, t_n) - u(x_j, t_n) \\
&\qquad + \mu(u(x_{j+1}, t_n) - 2u(x_j, t_n) + u(x_{j-1}, t_n))\Big| \\
&= \Big|(1 - 2\mu)\Big(U_j^n - u(x_j, t_n)\Big) + \mu\Big(U_{j+1}^n - u(x_{j+1}, t_n)\Big) \\
&\qquad + \mu\Big(U_{j-1}^n - u(x_{j-1}, t_n)\Big) - kT(x_j, t_n)\Big| \\
&\leq |1 - 2\mu| \cdot |e_j^n| + \mu|e_{j+1}^n| + \mu|e_{j-1}^n| + k|T(x_j, t_n)|,
\end{aligned}$$

where we used the triangle inequality.

If $\mu \leq \frac{1}{2}$ then $|1 - 2\mu| = 1 - 2\mu$, so

$$|e_j^{n+1}| \quad \leq \quad \max_j |e_j^n| + kM\left(\frac{k}{2} + \frac{h^2}{12}\right) \quad = \quad \max_j |e_j^n| + \frac{k^2}{2}M\left(1 + \frac{1}{6\mu}\right).$$

Now use induction. We have $e_i^0 = 0$, so

$$\begin{aligned}
|e_j^1| &\leq \frac{k^2}{2}M\left(1 + \frac{1}{6\mu}\right), \\
|e_j^2| &\leq 2\frac{k^2}{2}M\left(1 + \frac{1}{6\mu}\right), \\
&\vdots \\
|e_j^n| &\leq n\frac{k^2}{2}M\left(1 + \frac{1}{6\mu}\right) = \frac{k}{2}t_n M\left(1 + \frac{1}{6\mu}\right).
\end{aligned}$$

So $|e_j^n| \to 0$ as $k \to 0$ for all j. Hence the method is convergent when $\mu \leq \frac{1}{2}$.[37]

[37] Note that case $\mu > \frac{1}{2}$.

since $u_t = -au_x$ and $u_{tt} = -au_{xt} = -a\frac{\partial}{\partial x}u_t = a^2 u_{xx}$.

The von Neumann stability analysis. The growth factor of Lax-Wendroff scheme is

$$\begin{aligned} g(\theta) &= 1 - \frac{\mu}{2}\left(e^{ih\xi} - e^{-ih\xi}\right) + \frac{\mu^2}{2}\left(e^{-ih\xi} - 2 + e^{ih\xi}\right) \\ &= 1 - \mu i \sin\theta - 2\mu^2 \sin^2(\theta/2). \end{aligned}$$

where again $\theta = h\xi$. Therefore we proceed with the following derivation

$$\begin{aligned} |g(\theta)|^2 &= \left(1 - 2\mu^2 \sin^2\frac{\theta}{2}\right)^2 + \mu^2 \sin^2\theta \\ &= 1 - 4\mu^2 \sin^2\frac{\theta}{2} + 4\mu^4 \sin^4\frac{\theta}{2} + 4\mu^2 \sin^2\frac{\theta}{2}\left(1 - \sin^2\frac{\theta}{2}\right) \\ &= 1 - 4\mu^2\left(1 - \mu^2\right)\sin^4\frac{\theta}{2} \\ &\le 1 - 4\mu^2\left(1 - \mu^2\right). \end{aligned}$$

We conclude that $|g(\theta)| \le 1$ if $\mu \le 1$, that is $\Delta t \le h/|a|$. If $\Delta t > h/|a|$, there are ξ's such that $g(\theta)| > 1$ and the scheme is unstable.

Lax-Wendroff scheme is second order accurate both in time and space. To show this, we investigate the local truncation error:

$$\begin{aligned} T(x,t) &= \frac{u(x, t+\Delta t) - u(x,t)}{\Delta t} - \frac{a\left(u(x+h,t) - u(x-h,t)\right)}{2h} \\ &\quad - \frac{a^2 \Delta t\left(u(x-h,t) - 2u(x,t) + u(x+h,t)\right)}{2h^2} \\ &= u_t + \frac{\Delta t}{2}\, l - a u_{xx} - \frac{a^2\Delta t}{2} u_{xx} + O((\Delta t)^2 + h^2) \\ &= O((\Delta t)^2 + h^2) \end{aligned}$$

مركزية مركزية

9.11.1 A finite difference method (CT-CT) for the second order wave equation.

$$\frac{U_j^{k+1} - 2U_j^k + U_j^{k-1}}{(\Delta t)^2} = a^2 \frac{U_{j-1}^k - 2U_j^k + U_{j+1}^k}{h^2} \tag{9.23}$$

The method is second order accurate both in time and space $((\Delta t)^2 + h^2)$. The CFL constraints of this method is $\Delta t \le \frac{h}{|a|}$ this will be verified through the following discussion.

The von Neumann analysis gives:

$$\frac{g - 2 + 1/g}{(\Delta t)^2} = a^2 \frac{e^{-ih\xi} - 2 + e^{ih\xi}}{h^2}.$$

Let $\mu = \frac{|a|\Delta t}{h}$. The equation above becomes:

$$g^2 - 2g + 1 = \mu^2 \left(-4\mu^2 \sin^2\theta\right) g,$$

or

$$g^2 - 2\left(1 - 4\mu^2 \sin^2\theta\right) g + 1 = 0,$$

where $\theta = h\xi/2$. Solve the equation above to get

$$g = 1 - 2\mu^2 \sin^2\theta \pm \sqrt{(1 - 2\mu^2 \sin^2\theta)^2 - 1}.$$

Note that $1 - 2\mu^2 \sin^2\theta \le 1$. If we also have $1 - 2\mu^2 \sin^2\theta < -1$, then for one of roots

$$g_1 = 1 - 2\mu^2 \sin^2\theta - \sqrt{(1 - 2\mu^2 \sin^2\theta)^2 - 1} < -1,$$

or $|g_1| > 1$ for some θ, thus the scheme is unstable.

In order to have a stable scheme, we should require that $1 - 2\mu^2 \sin^2\theta \ge -1$, or $\mu^2 \sin^2\theta \le 1$. This can be guaranteed if $\mu^2 \le 1$ or $\Delta t \le h/|a|$. This is the CFL condition that we have expected. Under this CFL constraints, we would have

$$|g|^2 = \left(1 - 2\mu^2 \sin^2\theta\right)^2 + \left(1 - \left(1 - 2\mu^2 \sin^2\theta\right)^2\right) = 1,$$

since the second part in the expression of g is imaginary. Thus the scheme is neutrally stable.

Finite Differencing of Hyperbolic PDE's

Consider a simple example of a hyperbolic partial differential (or wave) equation with one spatial independent variable

$$u^2 \frac{\partial^2 \Phi}{\partial x^2} = \frac{\partial^2 \Phi}{\partial t^2}, \tag{23}$$

where u is the speed of the wave. The equivalent finite difference approximation is

$$u^2 \frac{\Phi(i-1,j) - 2\Phi(i,j) + \Phi(i+1,j)}{(\Delta x)^2} = \frac{\Phi(i,j+1) - 2\Phi(i,j) + \Phi(i,j-1)}{(\Delta t)^2}, \tag{24}$$

where $x = i\Delta x$, $i = 1,2,3,\ldots,n$, $t = j\Delta t$, $j = 1,2,\ldots$. In (24), we use the central difference

$$= \left(\frac{u\Delta t}{\Delta x} \right). \quad (25)$$

[illegible]

$$\boxed{\Phi(i,j+1) = 2(1-r)\Phi(i,j) + r[\Phi(i+1,j) + \Phi(i-1,j)] - \Phi(i,j-1)}. \tag{26}$$

[illegible]

$$\Phi(i,j+1) = \Phi(i+1,j) + \Phi \ldots$$ [illegible]

We assume a trial solution of the form

$$\Phi_i^n = A^n e^{jkix}. \tag{28}$$

Substituting this into Eq. (26) results in

$$A^{n+1} e^{jkix} = 2(1-r)A^n e^{jkix} + r(e^{jkx} + e^{-jkx})A^n e^{jkix} - A^{n-1} e^{jkix} \tag{29}$$

or

$$A^{n+1} = A^n \Big[2(1-r) + 2r\cos kx \Big] - A^{n-1}. \tag{30}$$

In terms of $g = A^{n+1}/A^n$, Eq. (30) becomes

$$g^2 - 2pg + 1 = 0 \tag{31}$$

where $p = 1 - 2r\sin^2 \frac{kx}{2}$. The quadratic equation (31) has solutions

$$g_1 = p + [p^2 - 1]^{1/2}, \qquad g_2 = p - [p^2 - 1]^{1/2}.$$

For $|g_i| \le 1$, where $i = 1,2$, p must lie between 1 and -1, i.e., $-1 \le p \le 1$ or

$$-1 \le 1 - 2r\sin^2 \frac{kx}{2} \le 1 \tag{32}$$

which implies that $r \le 1$ or $u\Delta t \le \Delta x$ for stability. This idea can be extended to show that the stability condition for two-dimensional wave equation is $u\Delta t/h < \frac{1}{\sqrt{2}}$, where $h = \Delta x = \Delta y$.

Simplification of von Neumann stability analysis for one step time marching method.

Assume that we have a one step time marching method $\mathbf{U}^{k+1} = f(\mathbf{U}^k, \mathbf{U}^{k+1})$. We have the following theorem that tells the stability of a finite difference method.

Theorem 8.2 *Let $\theta = h\xi$. A one-step finite difference scheme (with constant coefficients) is stable if and only if there is a constant K (independent of θ, Δt, and h) and some positive grid spacing Δt_0 and h_0 such that*

$$|g(\theta, \Delta t, h)| \le 1 + K\Delta t \tag{8.28}$$

for all θ, $0 < k \le k_0$, and $0 < h \le h_0$. If $g(\theta, \Delta t, h)$ is independent of h and Δt, the stability condition (8.28) can be replaced with

$$|g(\theta| \le 1. \tag{8.29}$$

This theorem shows that to determine the stability of a finite difference scheme we need consider only the amplification factor $g(h\xi) = g(\theta)$. This observation is due to von Neumann, and because of that, this analysis is usually called von Neumann analysis. We present some examples below.

We can follow the following steps for the von Neumann analysis:

1. Set $U_j^k = e^{ijh\xi}$. Plugging the expression into the finite difference scheme.
2. Express U_j^{k+1} as $U_j^{k+1} = g(\xi)e^{ijh\xi}$.
3. Solve $g(\xi)$ and determine whether or when $|g(\xi)| \le 1$ for the stability.

Example 2.

The stability of backward Euler's method for the heat equation $u_t = \beta u_{xx}$ is:

$$U_i^{k+1} = U_i^k + \mu\left(U_{i-1}^{k+1} - 2U_i^{k+1} + U_{i+1}^{k+1}\right), \quad \mu = \frac{\beta\Delta t}{h^2}.$$

Follow the procedure mentioned above, we have

$$\begin{aligned} g(\xi)e^{ijh\xi} &= e^{ijh\xi} + \mu\left(e^{i\xi(j-1)h} - 2e^{i\xi jh} + e^{i\xi(j+1)h}\right)g(\xi) \\ &= e^{i\xi jh}\left(1 + \mu\left(e^{-i\xi h} - 2 + e^{i\xi h}\right)g(\xi)\right). \end{aligned}$$

We solve $g(\xi)$ to get

$$\begin{aligned} g(\xi) &= \frac{1}{1 - \mu(e^{-i\xi h} - 2 + e^{i\xi h})} \\ &= \frac{1}{1 - \mu(2\cos(h\xi) - 2)} = \frac{1}{1 + 4\mu\sin^2(h\xi)/2} < 1 \end{aligned}$$