Numerical Integration

Basic Numerical Integration

- Trapezoidal Rule
- Simpson’s Rule
  - 1/3 Rule
  - 3/8 Rule
- Midpoint
- Gaussian Quadrature

Basic Numerical Integration

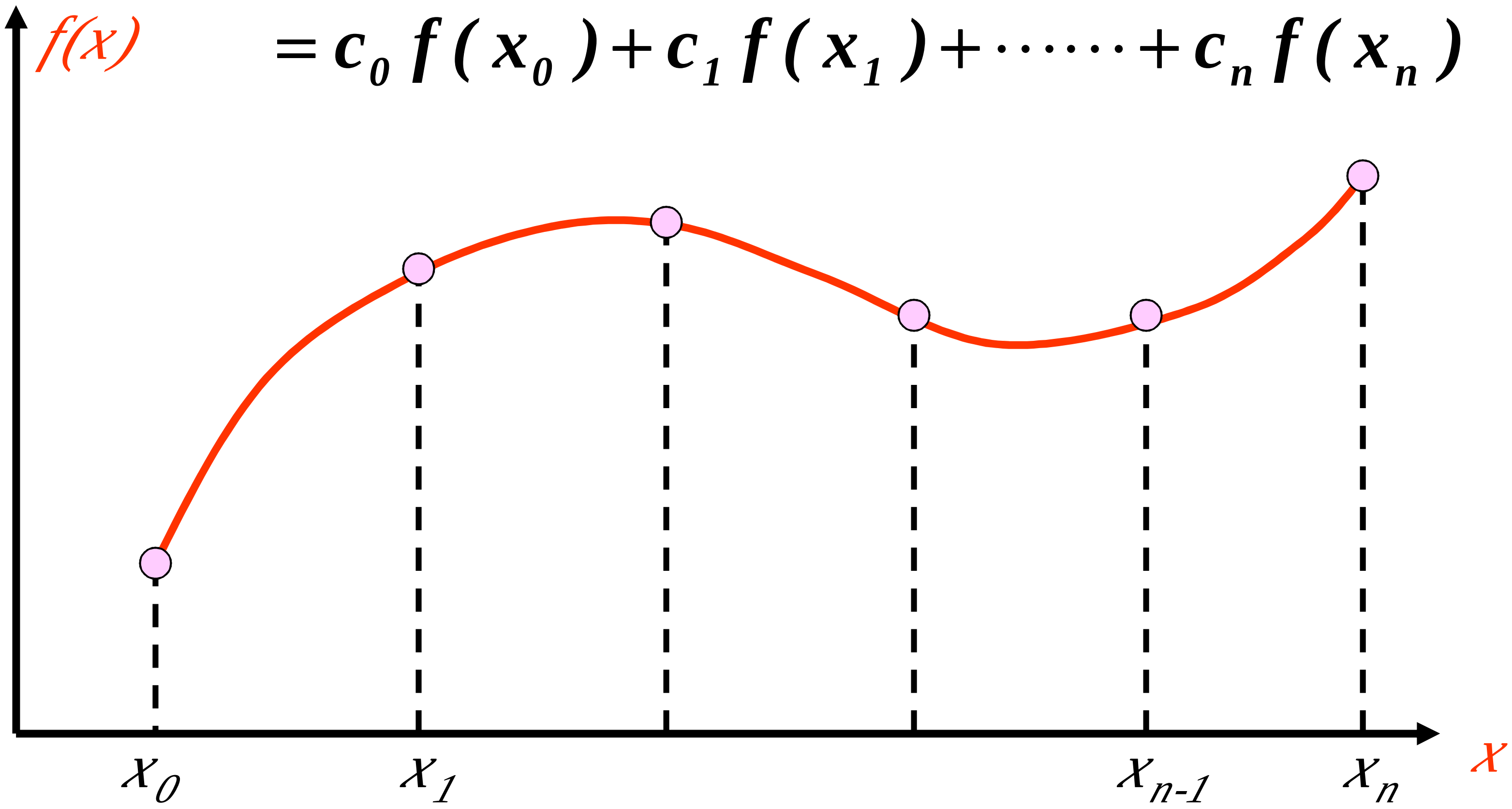
We want to find integration of functions of various forms of the equation known as the Newton Cotes integration formulas.

Basic Numerical Integration

- **Weighted sum of function values**

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} c_i f(x_i)$$

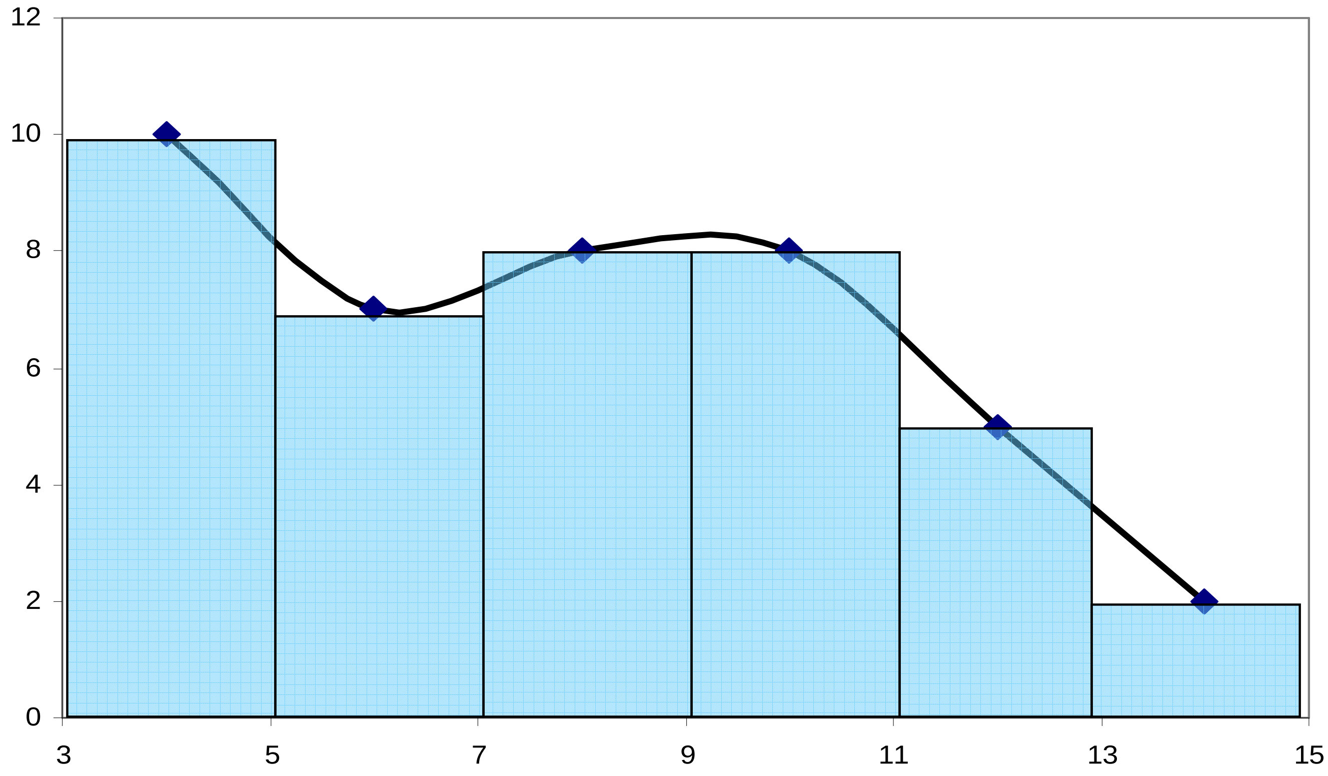
$$= c_0 f(x_0) + c_1 f(x_1) + \cdots\cdots + c_n f(x_n)$$



Numerical Integration

Idea is to do integral in small parts, like the way you first learned integration - a summation

Numerical methods just try to make it faster and more accurate



Numerical Integration

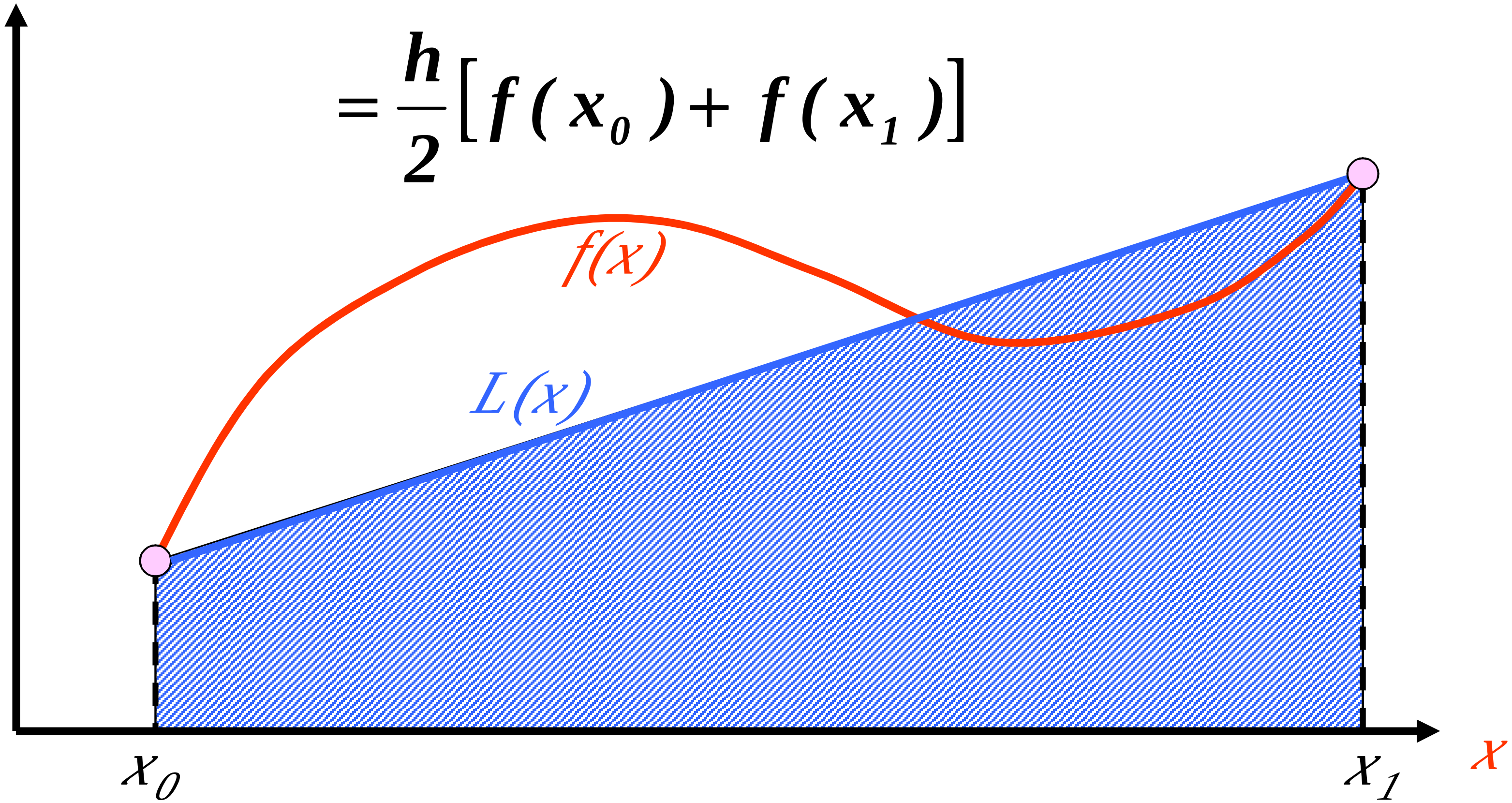
- **Newton-Cotes Closed Formulae -- Use both end points**
  - **Trapezoidal Rule : Linear**
  - **Simpson’s 1/3-Rule : Quadratic**
  - **Simpson’s 3/8-Rule : Cubic**
  - **Boole’s Rule : Fourth-order**
- **Newton-Cotes Open Formulae -- Use only interior points**
  - **midpoint rule**

Trapezoid Rule

- **Straight-line approximation**

$$\int_a^b f(x)dx \approx \sum_{i=0}^{1} c_i f(x_i) = c_0 f(x_0) + c_1 f(x_1)$$

$$= \frac{h}{2}\left[f(x_0) + f(x_1)\right]$$



Trapezoid Rule

- **Lagrange interpolation**

$$L(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

$$let \quad \mathrm{a} = \mathrm{x}_0,\ \mathrm{b} = \mathrm{x}_1,\ \ \xi = \frac{x - a}{b - a},\ \ \mathrm{d}\xi = \frac{\mathrm{dx}}{\mathrm{h}};\ \ h = b - a$$

$$\left\{\begin{matrix} x = a & \Rightarrow \xi = 0 \\ x = b & \Rightarrow \xi = 1 \end{matrix}\right\} \Rightarrow L(\xi) = (1 - \xi) f(a) + (\xi) f(b)$$

Trapezoid Rule

- **Integrate to obtain the rule**

$$\int_a^b f(x)dx \approx \int_a^b L(x)dx = h\int_0^1 L(\xi)d\xi$$

$$= f(a)h\int_0^1 (1-\xi)d\xi + f(b)h\int_0^1 \xi d\xi$$

$$= f(a)h(\xi - \frac{\xi^2}{2})\Big|_0^1 + f(b)h\frac{\xi^2}{2}\Big|_0^1 = \frac{h}{2}[f(a)+f(b)]$$

Example:Trapezoid Rule

Evaluate the integral $\int_0^4 xe^{2x}dx$

- **Exact solution**

$$\int_0^4 xe^{2x}dx = \left[\frac{x}{2}e^{2x} - \frac{1}{4}e^{2x}\right]_0^4$$

$$= \frac{1}{4}e^{2x}(2x-1)\Big|_0^1 = 5216.926477$$

- **Trapezoidal Rule**

$$I = \int_0^4 xe^{2x}dx \approx \frac{4-0}{2}\left[f(0)+f(4)\right] = 2(0+4e^8) = 23847.66$$

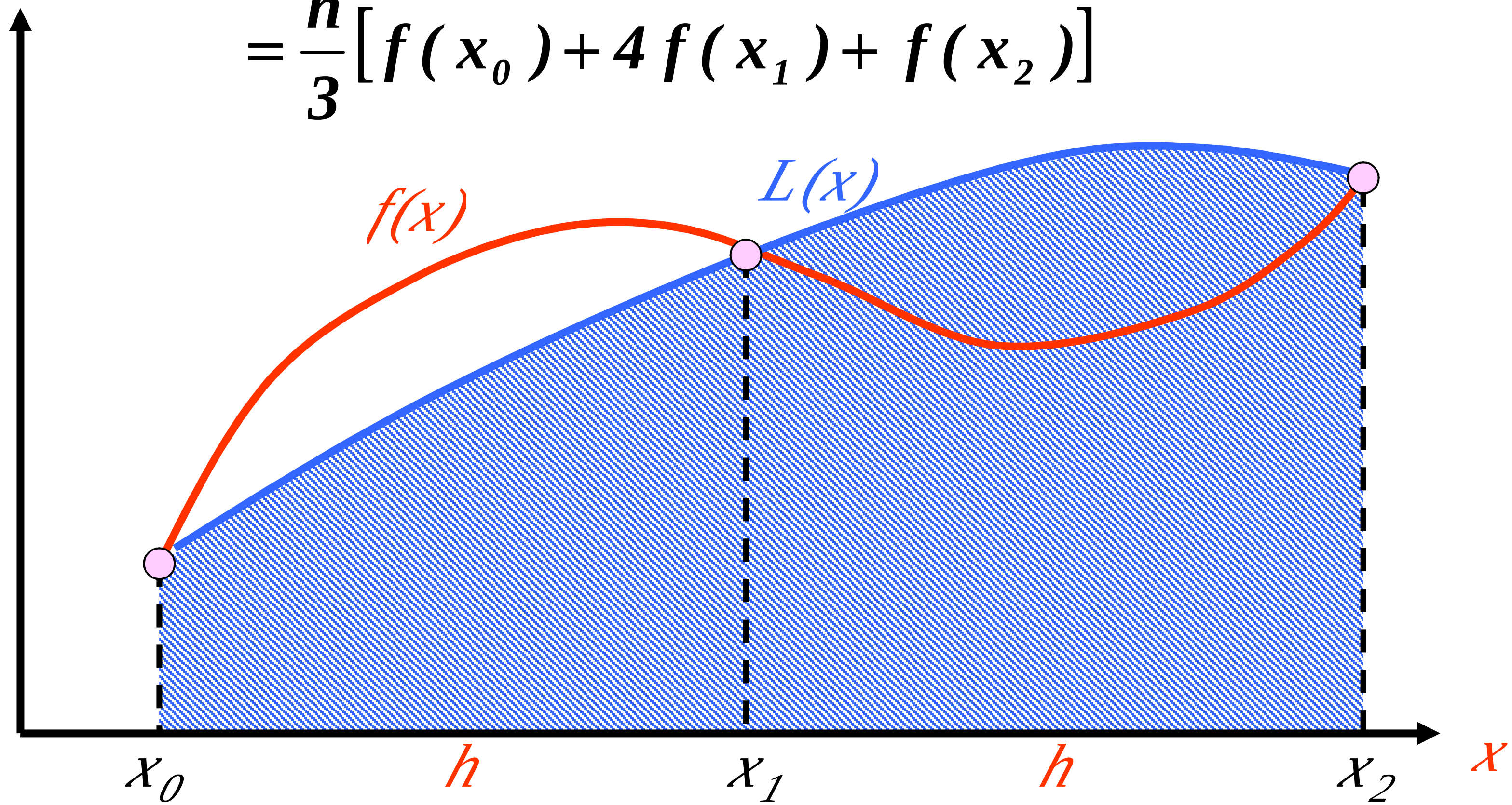
$$\varepsilon = \frac{5216.926 - 23847.66}{5216.926} = -357.12\%$$

Simpson's 1/3-Rule

Approximate the function by a parabola

$$\int_a^b f(x)dx \approx \sum_{i=0}^{2} c_i f(x_i) = c_0 f(x_0) + c_1 f(x_1) + c_2 f(x_2)$$

$$= \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$



Simpson's 1/3-Rule

$$L(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0)+\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1)$$

$$+\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2)$$

$$\text{let}\quad x_0=a,\ x_2=b,\ x_1=\frac{a+b}{2}$$

$$h=\frac{b-a}{2},\ \xi=\frac{x-x_1}{h},\ d\xi=\frac{dx}{h}$$

$$\begin{cases} x=x_0\Rightarrow\xi=-1\\ x=x_1\Rightarrow\xi=0\\ x=x_2\Rightarrow\xi=1\end{cases}$$

$$\boxed{L(\xi)=\frac{\xi(\xi-1)}{2}f(x_0)+(1-\xi^2)f(x_1)+\frac{\xi(\xi+1)}{2}f(x_2)}$$

Simpson's 1/3-Rule

Integrate the Lagrange interpolation

$$\int_a^b f(x)dx \approx h\int_{-1}^{1} L(\xi)d\xi = f(x_0)\frac{h}{2}\int_{-1}^{1}\xi(\xi-1)d\xi$$

$$+ f(x_1)h\int_0^1(1-\xi^2)d\xi + f(x_2)\frac{h}{2}\int_{-1}^{1}\xi(\xi+1)d\xi$$

$$= f(x_0)\frac{h}{2}\left(\frac{\xi^3}{3}-\frac{\xi^2}{2}\right)\Bigg|_{-1}^{1} + f(x_1)h\left(\xi-\frac{\xi^3}{3}\right)\Bigg|_{-1}^{1}$$

$$+ f(x_2)\frac{h}{2}\left(\frac{\xi^3}{3}+\frac{\xi^2}{2}\right)\Bigg|_{-1}^{1}$$

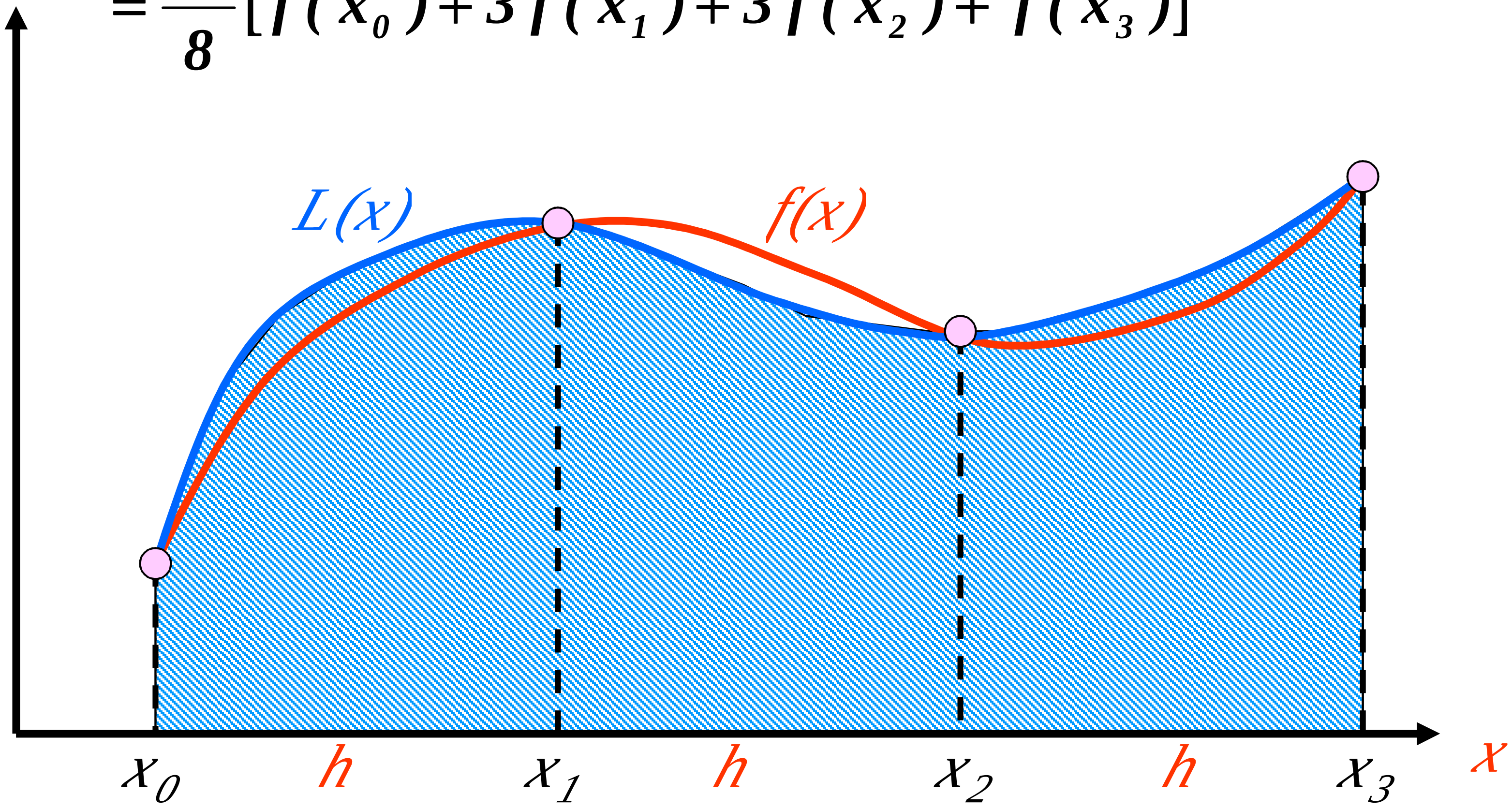
$$\boxed{\int_a^b f(x)dx = \frac{h}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right]}$$

Simpson's 3/8-Rule

Approximate by a cubic polynomial

$$\int_a^b f(x)dx \approx \sum_{i=0}^{3} c_i f(x_i) = c_0 f(x_0) + c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3)$$

$$= \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3) \right]$$



Simpson's 3/8-Rule

$$L(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0)$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

$$\int_a^b f(x)dx \approx \int_a^b L(x)dx \quad ; \quad h = \frac{b-a}{3}$$

$$= \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3) \right]$$

Example: Simpson's Rules

Evaluate the integral $\int_0^4 xe^{2x}dx$

- **Simpson's 1/3-Rule**

$$I = \int_0^4 xe^{2x}dx \approx \frac{h}{3}[f(0) + 4f(2) + f(4)]$$

$$= \frac{2}{3}\left[0 + 4(2e^4) + 4e^8\right] = 8240.411$$

$$\varepsilon = \frac{5216.926 - 8240.411}{5216.926} = -57.96\%$$

- **Simpson's 3/8-Rule**

$$I = \int_0^4 xe^{2x}dx \approx \frac{3h}{8}\left[f(0) + 3f(\frac{4}{3}) + 3f(\frac{8}{3}) + f(4)\right]$$

$$= \frac{3(4/3)}{8}[0 + 3(19.18922) + 3(552.33933) + 11923.832] = 6819.209$$

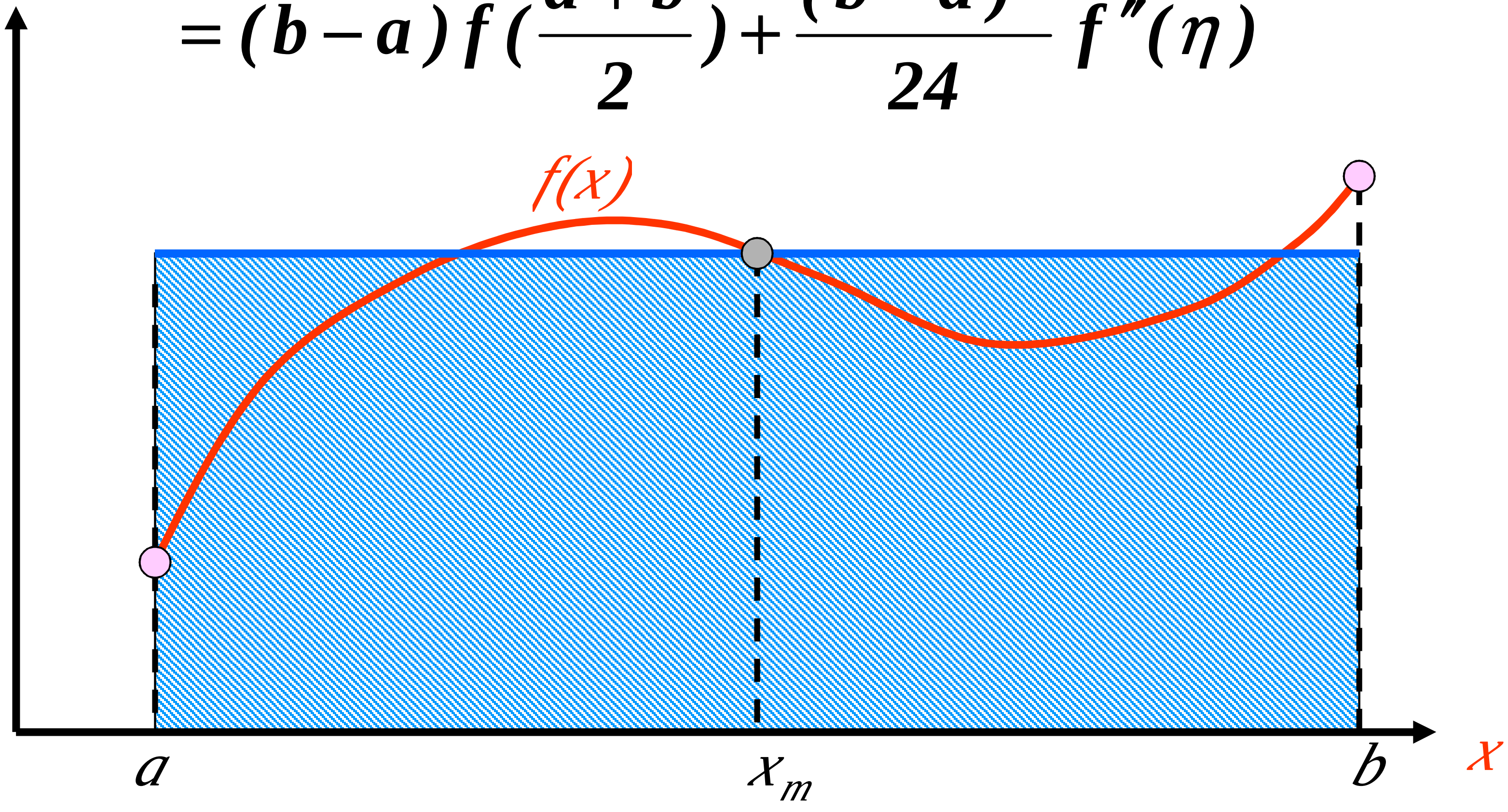
$$\varepsilon = \frac{5216.926 - 6819.209}{5216.926} = -30.71\%$$

Midpoint Rule

Newton-Cotes Open Formula

$$\int_a^b f(x)dx \approx (b-a)f(x_m)$$

$$= (b-a)f\left(\frac{a+b}{2}\right) + \frac{(b-a)^3}{24} f''(\eta)$$

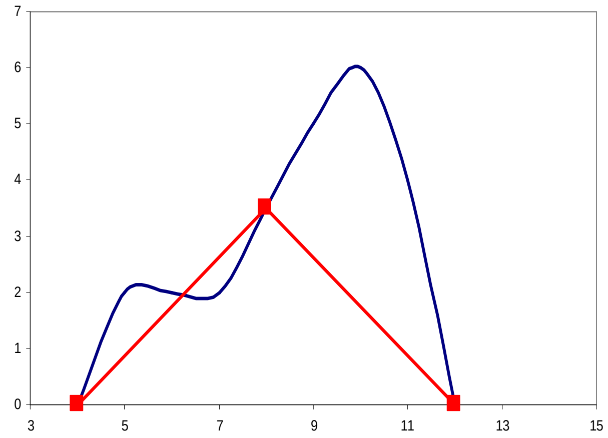


Better Numerical Integration

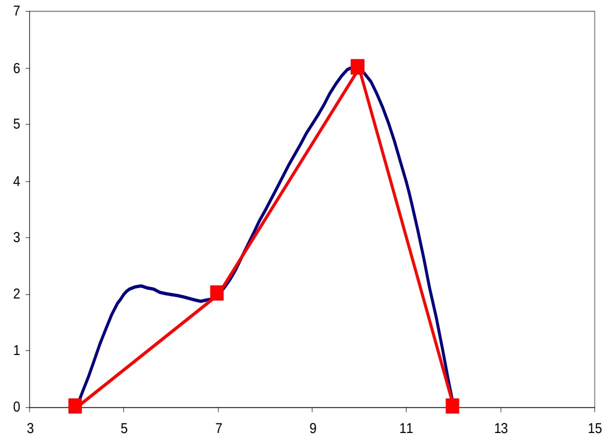
- **Composite integration**
  - **Composite Trapezoidal Rule**
  - **Composite Simpson's Rule**
- **Richardson Extrapolation**
- **Romberg integration**

Apply trapezoid rule to multiple segments over integration limits

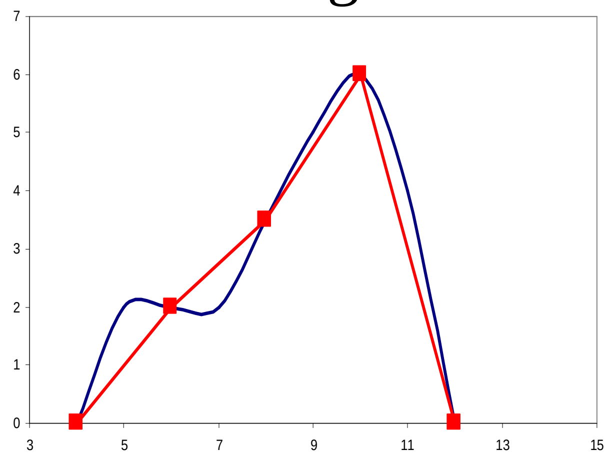
Two segments



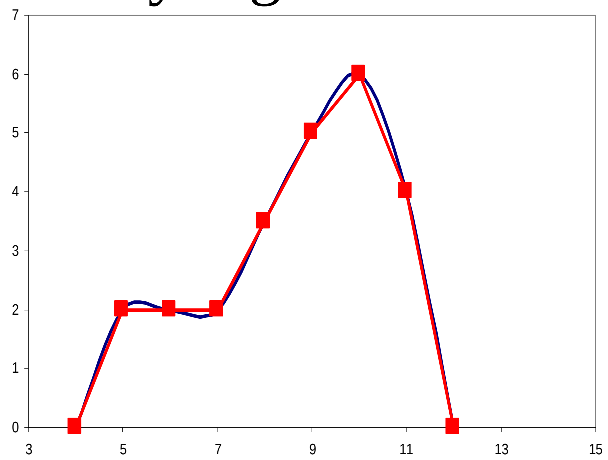
Three segments



Four segments



Many segments



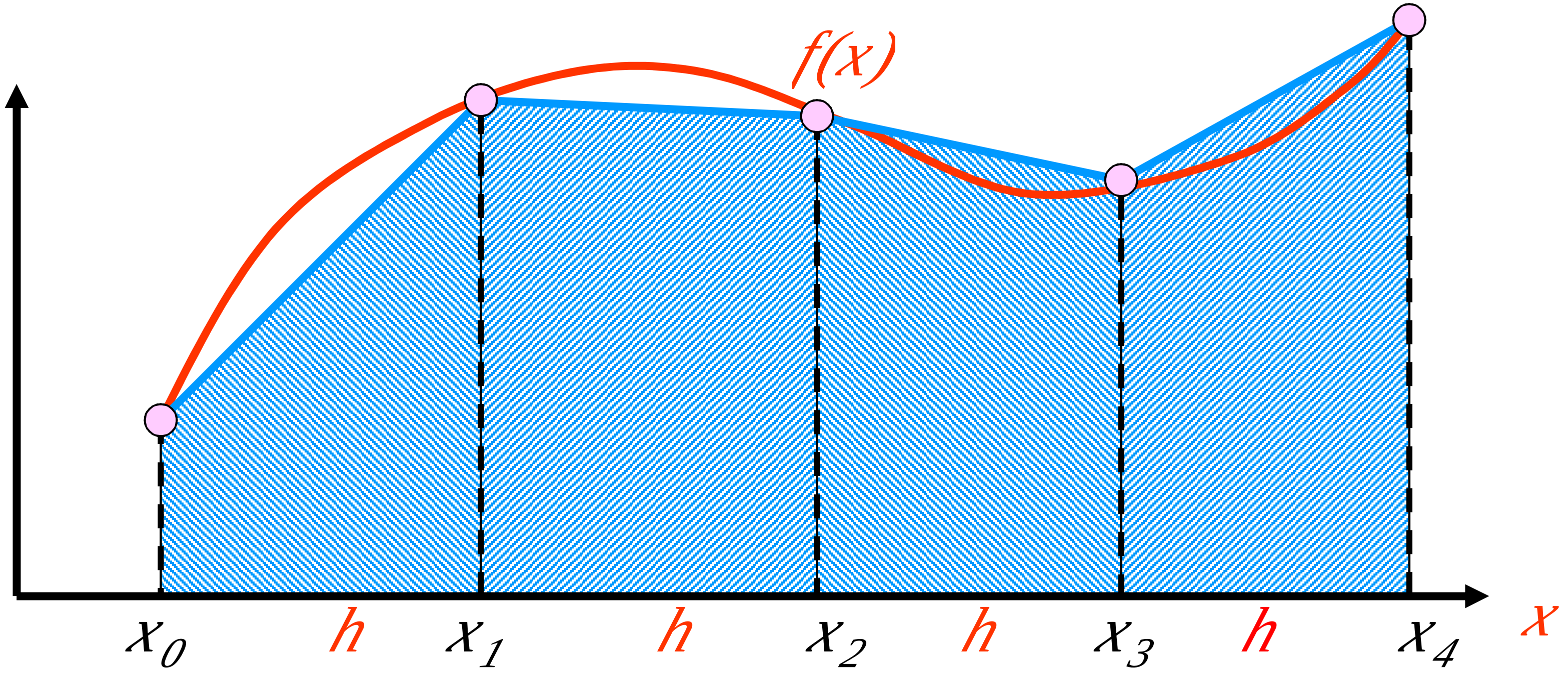
Composite Trapezoid Rule

$$\int_a^b f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots\cdots + \int_{x_{n-1}}^{x_n} f(x)dx$$

$$= \frac{h}{2}\left[f(x_0) + f(x_1)\right] + \frac{h}{2}\left[f(x_1) + f(x_2)\right] + \cdots + \frac{h}{2}\left[f(x_{n-1}) + f(x_n)\right]$$

$$= \frac{h}{2}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_i) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

$$h = \frac{b-a}{n}$$



Composite Trapezoid Rule

Evaluate the integral $$I = \int_0^4 xe^{2x}dx$$

$$n = 1, h = 4 \Rightarrow I = \frac{h}{2}\left[f(0) + f(4)\right] = 23847.66 \qquad \varepsilon = -357.12\%$$

$$n = 2, h = 2 \Rightarrow I = \frac{h}{2}\left[f(0) + 2f(2) + f(4)\right] = 12142.23 \qquad \varepsilon = -132.75\%$$

$$n = 4, h = 1 \Rightarrow I = \frac{h}{2}\left[f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)\right] = 7288.79 \qquad \varepsilon = -39.71\%$$

$$n = 8, h = 0.5 \Rightarrow I = \frac{h}{2}\left[f(0) + 2f(0.5) + 2f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + 2f(3) + 2f(3.5) + f(4)\right] = 5764.76 \qquad \varepsilon = -10.50\%$$

$$n = 16, h = 0.25 \Rightarrow I = \frac{h}{2}\left[f(0) + 2f(0.25) + 2f(0.5) + \cdots + 2f(3.5) + 2f(3.75) + f(4)\right] = 5355.95 \qquad \varepsilon = -2.66\%$$

Composite Trapezoid

Example

$$\int_1^2 \frac{1}{1+x}\,dx$$

x	f(x)
1.00	0.5000
1.25	0.4444
1.50	0.4000
1.75	0.3636
2.00	0.3333

Composite Trapezoid Rule with Unequal Segments

Evaluate the integral $I = \int_0^4 xe^{2x} dx$

- $h_1 = 2, h_2 = 1, h_3 = 0.5, h_4 = 0.5$

$$I = \int_0^2 f(x)dx + \int_2^3 f(x)dx + \int_3^{3.5} f(x)dx + \int_{3.5}^4 f(x)dx$$

$$= \frac{h_1}{2}[f(0) + f(2)] + \frac{h_2}{2}[f(2) + f(3)]$$

$$+ \frac{h_3}{2}[f(3) + f(3.5)] + \frac{h_4}{2}[f(3.5) + f(4)]$$

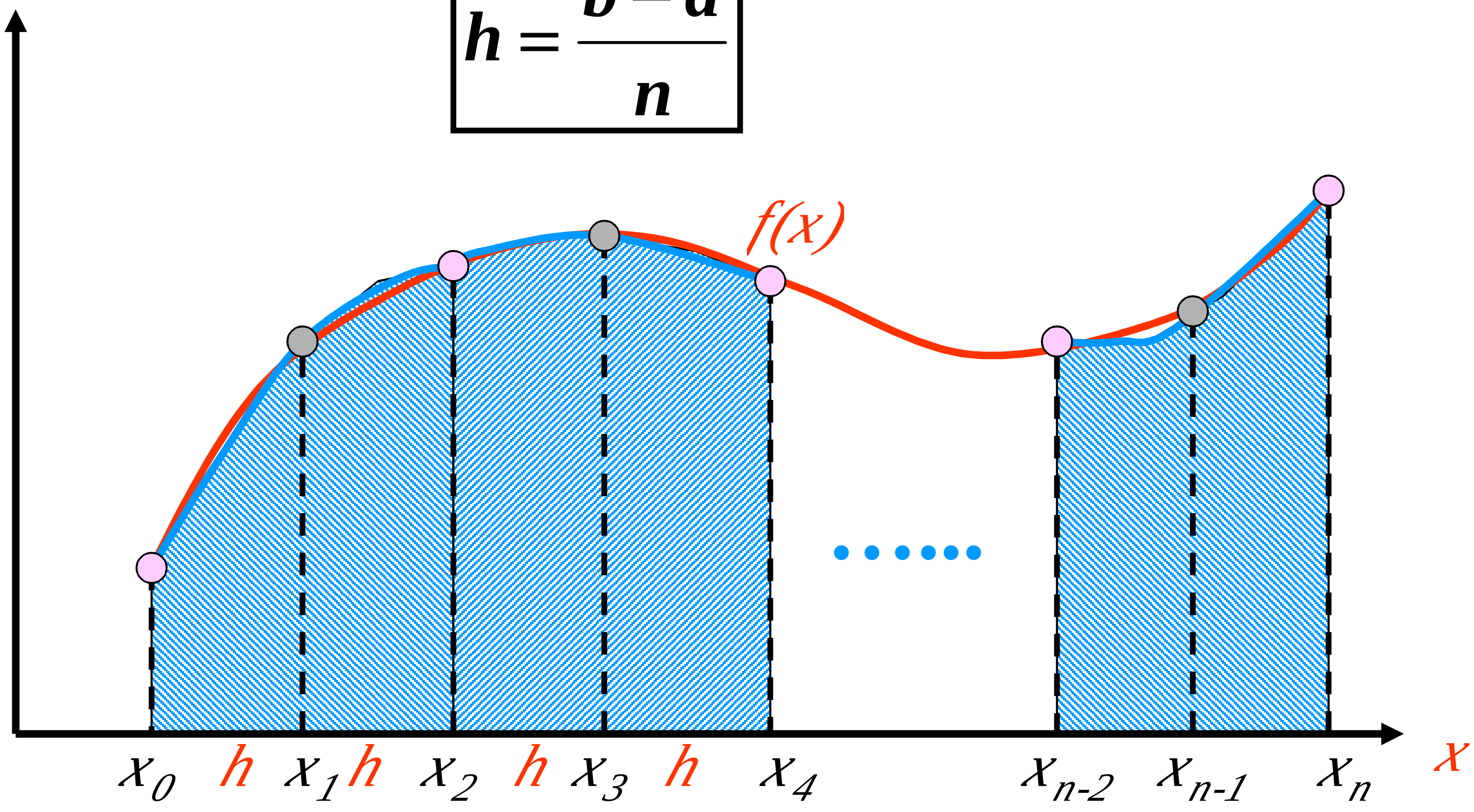
$$= \frac{2}{2}\left[0 + 2e^4\right] + \frac{1}{2}\left[2e^4 + 3e^6\right] + \frac{0.5}{2}\left[3e^6 + 3.5e^7\right]$$

$$+ \frac{0.5}{2}\left[3.5e^7 + 4e^8\right] = 5971.58 \qquad \Rightarrow \varepsilon = -14.45\%$$

Composite Simpson's Rule

Piecewise Quadratic approximations

$$h = \frac{b-a}{n}$$



Composite Simpson's Rule

Multiple applications of Simpson's rule

$$\int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)dx$$

$$= \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right] + \frac{h}{3}\left[f(x_2) + 4f(x_3) + f(x_4)\right]$$

$$+ \cdots + \frac{h}{3}\left[f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

$$= \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots\right.$$

$$+ 4f(x_{2i-1}) + 2f(x_{2i}) + 4f(x_{2i+1}) + \cdots$$

$$\left. + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

Composite Simpson's Rule

Evaluate the integral $I = \int_0^4 xe^{2x}dx$

- $n = 2, h = 2$

$$I = \frac{h}{3}\left[f(0) + 4f(2) + f(4) \right]$$
$$= \frac{2}{3}\left[0 + 4(2e^4) + 4e^8 \right] = 8240.411 \quad \Rightarrow \varepsilon = -57.96\%$$

- $n = 4, h = 1$

$$I = \frac{h}{3}\left[f(0) + 4f(1) + 2f(2) + 4f(3) + f(4) \right]$$
$$= \frac{1}{3}\left[0 + 4(e^2) + 2(2e^4) + 4(3e^6) + 4e^8 \right]$$
$$= 5670.975 \quad \Rightarrow \varepsilon = -8.70\%$$

Composite Simpson's Example

$$\int_1^2 \frac{1}{1+x}\,dx$$

x	f(x)
1.00	0.5000
1.25	0.4444
1.50	0.4000
1.75	0.3636
2.00	0.3333

Composite Simpson's Rule with Unequal Segments

Evaluate the integral $I = \int_0^4 xe^{2x}dx$

- $h_1 = 1.5, h_2 = 0.5$

$$I = \int_0^3 f(x)dx + \int_3^4 f(x)dx$$

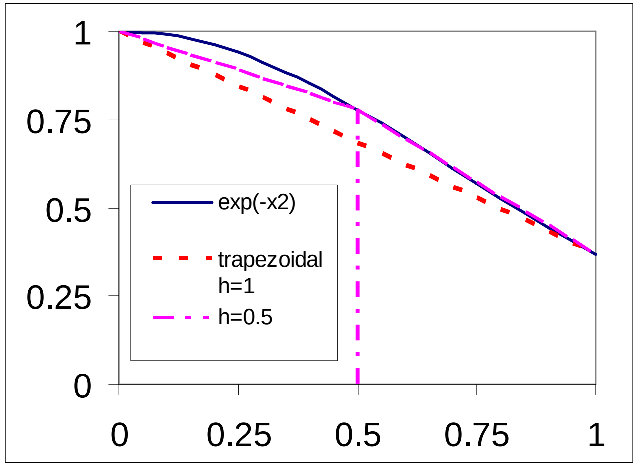
$$= \frac{h_1}{3}[f(0) + 4f(1.5) + 2f(3)]$$

$$+ \frac{h_2}{3}[f(3) + 4f(3.5) + 2f(4)]$$

$$= \frac{1.5}{3}\left[0 + 4(1.5e^3) + 3e^6\right] + \frac{0.5}{3}\left[3e^6 + 4(3.5e^7) + 4e^8\right]$$

$$= 5413.23 \quad \Rightarrow \varepsilon = -3.76\%$$

Example $f(x) = e^{-x^2}$



Trapezoidal

$$T = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n) \right]$$

$$h = 1 \quad \rightarrow \quad T = \frac{1}{2}[1 + 0.368] = 0.684$$

$$h = 0.5 \quad \rightarrow \quad T = \frac{0.5}{2}[1 + 2(0.779) + 0.368] = 0.731$$

$$h = 0.25 \quad \rightarrow \quad T = \frac{0.25}{2}[1 + 2(0.939) + 2(0.779) + 2(0.570) + 0.368] = 0.743$$

Simpson's

$$S = \frac{h}{3}\left[f(x_0) + 4\sum_{i=1,3..}^{n-1} f(x_i) + 2\sum_{i=2,4..}^{n-2} f(x_i) + f(x_n) \right]$$

$$h = 0.5 \quad \rightarrow \quad S = \frac{0.5}{3}[1 + 4(0.779) + 0.368] = 0.7472$$

$$h = 0.25 \quad \rightarrow \quad S = \frac{0.25}{3}[1 + 4(0.939) + 2(0.779) + 4(0.570) + 0.368] = 0.7469$$

Example 7 Calculate the integral

$$\int_0^1 e^{x^2} dx$$

by using trapezoid rule, Simpson's rule, composite trapezoid rule and Simpson's rule with tw subintervals.

Solution By trapezoid rule

$$\int_0^1 e^{x^2} dx \approx \frac{1}{2}(1+e) = 1.859140914\,.$$

By Simpson's rule,

$$\int_0^1 e^{x^2} dx \approx \frac{1}{6}(1+4*e^{1/4}+e) = 1.475730583\,.$$

By trapezoid rule with two subintervals, $h = 1/2$

$$\int_0^1 e^{x^2} dx \approx \frac{1/2}{2}(1+2e^{1/4}+e) = 1.571583165\,.$$

By Simpson's rule with two subintervals, $h = 1/2$,

$$\int_0^1 e^{x^2} dx \approx \frac{1/2}{6}(1+4e^{1/16}+2e^{1/4}+4e^{9/16}+e) = 1.46371076\,.$$

The exact solution is 1.46265177159149.

The error for Newton Cotes formulas

- The error for Newton Cotes formulas can be given by

$$\text{error} = \frac{1}{(n+1)!}\int_a^b \prod_{i=0}^{n}(x - x_i) f^{(n+1)}(\xi)\,dx$$

Trapezoid error analysis

$$f(x) = P_1(x) + \frac{(x-a)(x-b)}{2} f''(\xi) \, .$$

Then

$$\int_a^b f(x)dx = \int_a^b P_1(x)dx + \int_a^b \frac{(x-a)(x-b)}{2} f''(\xi)dx$$

By the Mean-Value theory,

$$\text{error} = \int_a^b f(x)dx - \frac{b-a}{2}(f(a)+f(b)) = \frac{-(b-a)^3}{12} f''(\eta)$$

Simpson's rule error analysis

$$\text{error} = \int_a^b f(x)dx - \frac{b-a}{6}\left(f(a) + 4f(\frac{b+a}{2}) + f(b)\right) = \int_a^b \frac{(x-a)(x-(a+b)/2)(x-b)}{6} f'''(\xi)dx$$

We can prove that

$$\int_a^b \frac{(x-a)(x-(a+b)/2)(x-b)}{6} f'''(\xi)dx = -\frac{1}{90}\left(\frac{b-a}{2}\right)^5 f^{(4)}(\eta).$$

Composite Newton-Cotes error analysis

- the error for composite trapezoid rule is

$$-\frac{1}{12} f^{(2)}(b-a)h^2$$

- the error for composite Simpson's rule is

$$-\frac{b-a}{180} h^4 f^{(4)}(c_x)$$

Composite-trapezoid error analysis

- The error of trapezoid applied to single

interval is $-\frac{1}{12} f^{(2)}(\xi)(b-a)^3$

If we apply the composite trapezoid to n, the error can be written as:

$$-\frac{1}{12} f^{(2)}(b-a)h^2$$

Composite Simpson's rule

- Given the interval [a,b], h=(b-a)/2 and find the quadratic interpolating polynomial passing through (a,f(a)),(a+h,f(a+h)) and (b,f(b)).

$$\frac{1}{3}\left(f(a) + 4f(a+h) + f(b) \right) h$$

$$error \approx -\frac{1}{90} f^{4}(\xi) h^{5}$$

- the error for composite Simpson's rule is

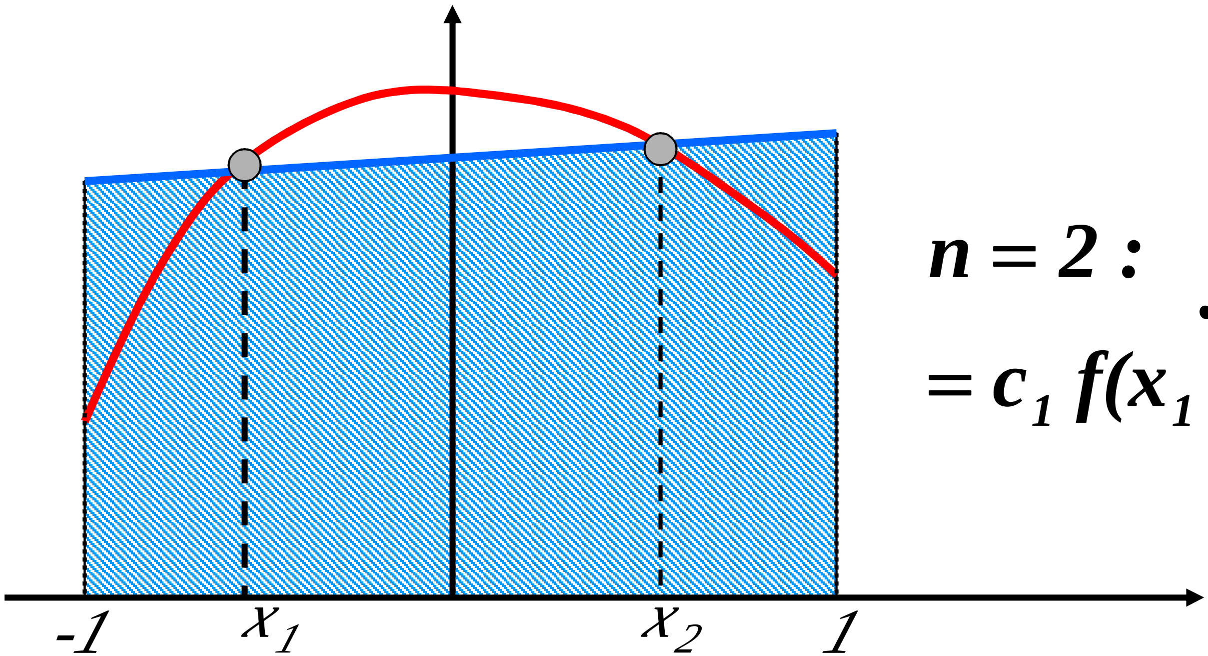
$$-\frac{b-a}{180} h^{4} f^{(4)}(c_x)$$

Gaussian Quadratures

- **<u>Newton-Cotes Formulae</u>**
  - use evenly-spaced functional values
- **<u>Gaussian Quadratures</u>**
  - select functional values at non-uniformly distributed points to achieve higher accuracy
  - change of variables so that the interval of integration is [-1,1]

Gaussian Quadrature on **[-1, 1]**

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} c_i f(x_i) = c_1 f(x_1) + c_2 f(x_2) + \cdots + c_n f(x_n)$$



$$n = 2: \int_{-1}^{1} f(x)dx = c_1 f(x_1) + c_2 f(x_2)$$

- *Choose* (c_1, c_2, x_1, x_2) **such that the method yields "exact integral" for** $f(x) = x^0, x^1, x^2, x^3$

Gaussian Quadrature on [-1, 1]

$$n = 2: \quad \int_{-1}^{1} f(x)dx = c_1 f(x_1) + c_2 f(x_2)$$

Exact integral for $f = x^0, x^1, x^2, x^3$

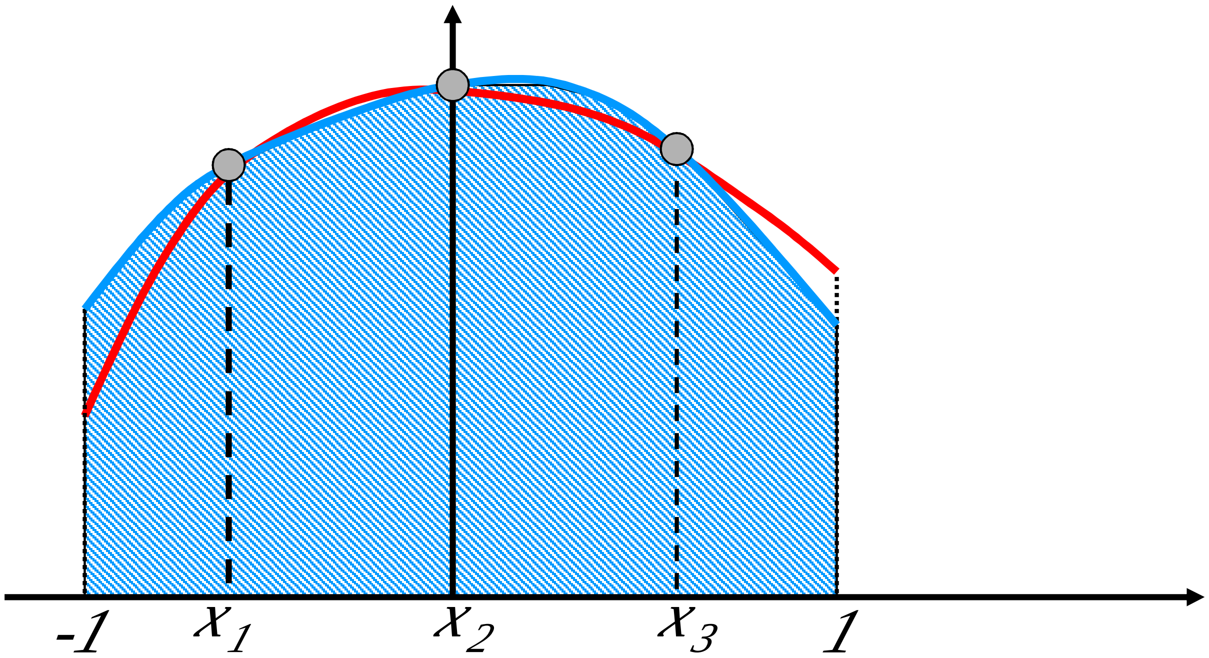
– **Four equations for four unknowns**

$$\begin{cases} f = 1 \quad \Rightarrow \int_{-1}^{1} 1dx = 2 = c_1 + c_2 \\ f = x \quad \Rightarrow \int_{-1}^{1} xdx = 0 = c_1 x_1 + c_2 x_2 \\ f = x^2 \Rightarrow \int_{-1}^{1} x^2 dx = \dfrac{2}{3} = c_1 x_1^2 + c_2 x_2^2 \\ f = x^3 \Rightarrow \int_{-1}^{1} x^3 dx = 0 = c_1 x_1^3 + c_2 x_2^3 \end{cases} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = 1 \\ x_1 = \dfrac{-1}{\sqrt{3}} \\ x_2 = \dfrac{1}{\sqrt{3}} \end{cases}$$

$$I = \int_{-1}^{1} f(x)dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

Gaussian Quadrature on **[-1, 1]**

$$n = 3: \int_{-1}^{1} f(x)dx = c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3)$$



- **Choose** $(c_1, c_2, c_3, x_1, x_2, x_3)$ **such that the method yields "exact integral" for** $f(x) = x^0, x^1, x^2, x^3, x^4, x^5$

Gaussian Quadrature on **[-1, 1]**

$$f = 1 \Rightarrow \int_{-1}^{1} x\,dx = 2 = c_1 + c_2 + c_3$$

$$f = x \Rightarrow \int_{-1}^{1} x\,dx = 0 = c_1 x_1 + c_2 x_2 + c_3 x_3$$

$$f = x^2 \Rightarrow \int_{-1}^{1} x^2\,dx = \frac{2}{3} = c_1 x_1^2 + c_2 x_2^2 + c_3 x_3^2$$

$$f = x^3 \Rightarrow \int_{-1}^{1} x^3\,dx = 0 = c_1 x_1^3 + c_2 x_2^3 + c_3 x_3^3$$

$$f = x^4 \Rightarrow \int_{-1}^{1} x^4\,dx = \frac{2}{5} = c_1 x_1^4 + c_2 x_2^4 + c_3 x_3^4$$

$$f = x^5 \Rightarrow \int_{-1}^{1} x^5\,dx = 0 = c_1 x_1^5 + c_2 x_2^5 + c_3 x_3^5$$

$$\Rightarrow \begin{cases} c_1 = 5/9 \\ c_2 = 8/9 \\ c_3 = 5/9 \\ x_1 = -\sqrt{3/5} \\ x_2 = 0 \\ x_3 = \sqrt{3/5} \end{cases}$$

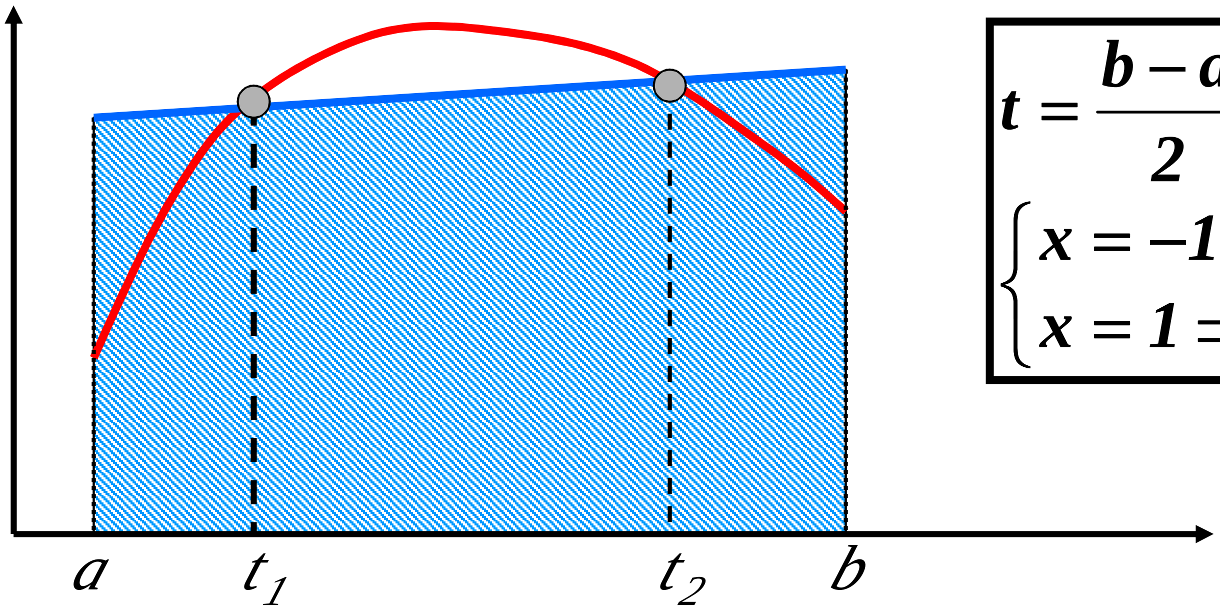
Gaussian Quadrature on [-1, 1]

Exact integral for $f = x^0, x^1, x^2, x^3, x^4, x^5$

$$I = \int_{-1}^{1} f(x)dx = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$

Gaussian Quadrature on [a, b]

Coordinate transformation from [a,b] to [-1,1]



$$t = \frac{b-a}{2}x + \frac{b+a}{2}$$

$$\begin{cases} x = -1 \Rightarrow t = a \\ x = 1 \Rightarrow \quad t = b \end{cases}$$

$$\int_a^b f(t)dt = \int_{-1}^1 f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)\left(\frac{b-a}{2}\right)dx = \int_{-1}^1 g(x)dx$$

Example: Gaussian Quadrature

Evaluate $I = \int_0^4 te^{2t}dt = 5216.926477$

Coordinate transformation

$$t = \frac{b-a}{2}x + \frac{b+a}{2} = 2x + 2; \quad dt = 2dx$$

$$I = \int_0^4 te^{2t}dt = \int_{-1}^{1}(4x+4)e^{4x+4}dx = \int_{-1}^{1} f(x)dx$$

Two-point formula

$$I = \int_{-1}^{1} f(x)dx = f(\frac{-1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}}) = (4 - \frac{4}{\sqrt{3}})e^{4-\frac{4}{\sqrt{3}}} + (4 + \frac{4}{\sqrt{3}})e^{4+\frac{4}{\sqrt{3}}}$$

$$= 9.167657324 + 3468.376279 = 3477.543936 \qquad (\varepsilon = 33.34\%)$$

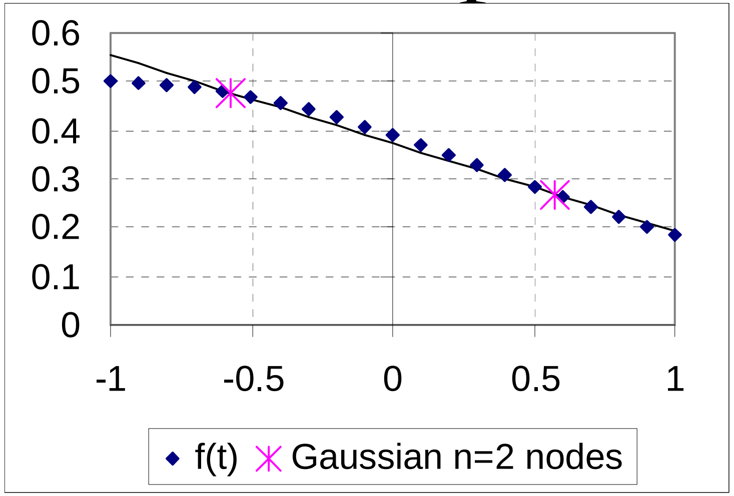
Gaussian Quadrature - Example

$$I = \int_0^1 e^{-x^2}\,dx \quad =$$

$$I = \int_a^b f(x)dx \quad = \int_{-1}^{1} f(t)dt$$

$$where \quad t = \frac{2x-(b+a)}{b-a} =$$

$$dx =$$



$$I \cong 1 \cdot f(t_0 = \frac{-1}{\sqrt{3}}) + 1 \cdot f(t_0 = \frac{1}{\sqrt{3}})$$

$$= 0.7466 \qquad \varepsilon_r = 0.03\%$$

Example: Gaussian Quadrature

Three-point formula

$$I = \int_{-1}^{1} f(x)dx = \frac{5}{9} f(-\sqrt{0.6}) + \frac{8}{9} f(0) + \frac{5}{9} f(\sqrt{0.6})$$

$$= \frac{5}{9}(4 - 4\sqrt{0.6})e^{4-\sqrt{0.6}} + \frac{8}{9}(4)e^{4} + \frac{5}{9}(4 + 4\sqrt{0.6})e^{4+\sqrt{0.6}}$$

$$= \frac{5}{9}(2.221191545) + \frac{8}{9}(218.3926001) + \frac{5}{9}(8589.142689)$$

$$= 4967.106689 \qquad (\varepsilon = 4.79\%)$$

Four-point formula

$$I = \int_{-1}^{1} f(x)dx = 0.34785\left[f(-0.861136) + f(0.861136)\right]$$

$$+ 0.652145\left[f(-0.339981) + f(0.339981)\right]$$

$$= 5197.54375 \qquad (\varepsilon = 0.37\%)$$

Summary

- Integration Techniques
  - **Trapezoidal Rule : Linear**
  - **Simpson’s 1/3-Rule : Quadratic**
  - **Simpson’s 3/8-Rule : Cubic**
  - **Boole’s Rule : Fourth-order**
- Gaussian Quadrature